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Elements of dynamics

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# ELEMENTS OF DYNAMICS.

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Second Edition.—WITH AN APPENDIX.

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# ELEMENTS OF DYNAMICS.

EDITED, WITH PERMISSION, BY

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FROM NOTES OF LECTURES DELIVERED BY ✓

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APPENDIX.

# INTRODUCTION.

§ 1. Dynamics is the Science which investigates the action of Force.

§ 2. Force is recognised as acting in two ways: 1° so as to compel rest or to prevent change of motion, and 2° so as to produce or to change motion. Dynamics, therefore, is divided into two parts, Statics and Kinetics.

§ 3. In Statics, the action of force in maintaining rest, or preventing change of motion, "the balancing of forces," is investigated; in Kinetics, the action of force in producing or in changing motion.

Remark.—In Kinetics it is not mere *motion* which is investigated, but the relation of *forces* to motion. The circumstances of mere motion, considered without reference to the bodies moved, or to the forces producing the motion, or to the forces called into action by the motion, constitute the subject of a branch of Pure Mathematics, which is called "Kinematics," or in its more practical branches, "Mechanism."

§ 4. Before the laws and actions of force can be investigated, it is necessary to know what is meant by the term Force. An explanation has been virtually given already in § 2. Force is involved in our conception of the "properties of matter," and is known only in and through phenomena presented by matter. Whenever two portions of matter tend in any way to occupy the same space, their mutual impenetrability prevents them from doing so, and force arises between them. If the force and the resistance be equal, rest may, as we shall see, be the result. If either force be greater than the other, the less will be overcome, and motion will be the result.

§ 5. The word "Force," is used, as will be observed, to denote the general conception of that which produces change of motion.

And it is employed also, in particular, to denote the cause involved in a particular case of rest or motion.

§ 6. In Statics, the action of force in balancing a tendency to motion, or to change of motion, is examined; its different cases classified; and the general principles laid down, which enable us to determine what will result under any circumstances of the action of any number of individual forces.

§ 7. The same thing is done in Kinetics for forces producing change of motion.

§ 8. In order to the clear comprehension of the cases which will be considered in Statics and Kinetics, some explanations under the three heads following will first be given:—

1°. What is meant by a material point and a rigid body?
2°. What the three elements which specify a force are;
3°. The method of representing forces as to direction and magnitude.

§ 9. 1°. A material point.—The difference between a mathematical and a material point, corresponds perfectly with the difference between pure and applied mathematics. The mathematical point is mere position; the material point is a point imagined to possess the capability of being acted on by force. In practice, the material point is always some finite mass of matter, either considered as a whole, on which the force acts, or of which some point is idealized for the purposes of investigation and calculation. The point may be a grain of sand, or a cannon ball, or the blade of an oar; or it may be the Moon, the Earth, the Sun, without affecting the nature of the question, provided always this does not involve the dimensions of the space through which the matter is spread. Thus, if a metal ring be in equilibrium when acted on by a number of forces, it may be regarded as a material point whether it be an engine fly-wheel, or a ring of $\frac{1}{1000}$ of an inch diameter, provided only, the lines of the forces all pass through one point, as for instance its centre. A collection of material points rigidly connected together, may be imagined as constituting a rigid body.

§ 10. 2°. The three elements specifying a force, or the three elements which must be known, before a clear notion of the force under consideration, can be formed, are, its place of application, its direction, and its magnitude.

§ 11. The place of application of a force. The first case to be considered is that in which the place of application is a point. It has been shewn already (§ 9) in what sense the term "point," is to be taken, and, therefore, in what way a force may be imagined as acting at a point. In reality, however, the place of application of a force, is always either a surface, or a space of three dimensions occupied by matter. The point of the finest needle, or the edge of the sharpest knife is still a surface, and acts as such on the bodies to which it may be applied. Even the most rigid substances, when brought together, do not touch at a point merely, but mould each other so as to produce a surface of application. On the other hand, gravity is a force of which the place of application is the whole matter of the body whose weight is considered; and the smallest particle of matter that has weight, occupies some finite portion of space.

§ 12. Thus it is to be remarked, that there are two kinds of force, distinguishable by their place of application—force, whose place of application is a surface, and force, whose place of application is a solid. When a heavy body rests on the ground or on a table, force of the second character, acting downwards, is balanced by force of the first character acting on the whole upwards.

§ 13. The second element in the specification of a force is its direction. The direction of a force' is the line in which it acts. If the place of application of a force be regarded as a point, a line through that point, in the direction in which the force tends to move the body, is the direction of the force. In the case of a force distributed over a surface, it is frequently possible and convenient to assume a single point and a single line, such that a certain force acting at that point in that line, would produce the same effect as is really produced.

§ 14. The third element in the specification of a force is its magnitude. This involves a consideration of the method followed in Dynamics for measuring forces. Before measuring anything it is necessary to have a unit of measurement, or a standard to which to refer, and a principle of numerical specification, or a mode of referring to the standard.

§ 15. The force we are most accustomed to measure is the weight of bodies, that is, the force of gravity upon them. Any other force may be measured by the force of gravity on the standard of measurement, or on a multiple or a proper or improper fraction of the standard, which it can exactly counterbalance. A force capable of supporting 20 tons is a 20 ton force, and this is the common method of reckoning. But the force of gravity and, therefore, the weight of bodies, varies at different points of the earth's surface, and so gravity, without a specified locality, furnishes no fixed and universal basis for measuring force by. If we do take a gravitation standard for force, we must indicate where on the earth's surface the measurement is made.

§ 16. The formula, deduced by Clairault from observation, and a certain theory regarding the figure and density of the earth, may be employed to calculate the most probable value of the apparent force of gravity, being the resultant of true gravitation and centrifugal force, in any locality where no pendulum observation of sufficient accuracy has been made. This formula, with the two co-efficients which it involves, corrected according to the best modern pendulum observations, is as follows:—

Let G be the apparent force of gravity at the equator, and $g$ at any latitude $\lambda$; then

$$g = G(1 + {\cdot}005133 \sin^2\lambda).$$

The value of G, in terms of the absolute unit to be explained immediately, is

$$32{\cdot}088.$$

According to this formula, therefore, polar gravity will be

$$g = 32{\cdot}088 \times 1{\cdot}005133 = 32{\cdot}2527.$$

§ 17. Gravity having failed to furnish a definite standard, independent of locality, recourse must be had to something else. The principle of measurement introduced by Gauss, furnishes us with what we want.

§ 18. According to this principle, the standard or unit force is that force which, acting on the unit of matter during the unit of time, generates the unit of velocity.

This is known as Gauss's absolute unit; absolute, because it furnishes a standard independent of the differing force of gravity at different localities.

§ 19. The absolute unit depends on the unit of matter, the unit of time, and the unit of velocity; and as the unit of velocity depends on the unit of space and the unit of time, there is, in the definition, a single reference to mass and space, but a *double* reference to time; and this is a point that must be particularly attended to.

§ 20. The unit of mass may be the British imperial pound; the unit of space the British standard foot; and the unit of time the mean solar second.

We accordingly define the British absolute unit force as "the force which, acting on one pound of matter for one second, generates a velocity of one foot per second."

§ 21. To render this standard intelligible, all that has to be done is to find how many absolute units will produce the same effect as the force of gravity. The way to do this is to measure the effect of gravity in producing acceleration on a body unresisted in any way. The most accurate method is indirect, by means of the pendulum. The result of pendulum experiments made at Leith Fort, by Captain Kater, is, that the velocity acquired by a body falling unresisted for one second, is at that place 32·207 feet per second. The preceding formula gives exactly 32·2, for the latitude, 55° 33′. This is about twenty nautical miles south of the Glasgow University Observatory, of which the latitude is 55° 52′ 43″. The variation in the force of gravity for one degree of difference of latitude about the latitude of Glasgow is only ·0000832 of its own amount. It is nearly the same, though somewhat more, for every degree of latitude southwards, as far as the southern limits of the British Isles. On the other hand, the variation per degree would be sensibly less, as far north as the Orkney and Shetland Isles. Hence the augmentation of gravity per degree from south to north throughout the British Isles is at most about $\frac{1}{12000}$ of its whole amount in any locality. The average for the whole of Great Britain and Ireland differs certainly but little from 32·2. Our present application is, that the force of gravity at Glasgow is 32·2 times the force which, acting on a pound for a second, would generate a velocity of one foot per second; in other words, 32·2 is the number of absolute units which measures the weight of a pound in this latitude.

§ 22. 3°. The method of representing forces as to their direction and magnitude by straight lines. A line is drawn in the direction of the force; any convenient point in it is chosen as a point of reference, which may or may not be the place of application of the force; and from that point any length is measured off along the line to represent the magnitude of the force. When there are more forces than one to be represented in the diagram, the lengths of the lines representing them must be proportional to the magnitudes of the forces. The direction of any force thus represented is denoted by an arrow-head; and, in reading the direction by letters, it is proper to begin with the letter from which the arrow-head is turned. Thus, let the straight line, B C (fig. 1), represent a force, P. The position of the arrow-head shows that P acts from B towards C, and its direction would therefore be signified by reading, B C. For the same reason the direction of the force, P (fig. 2), would be signified by reading, C B.

Fig. 1.

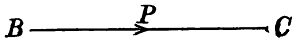
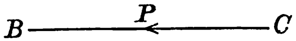
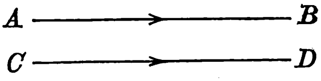


Fig. 2.



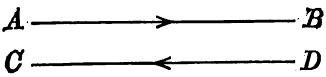
§ 23. Forces are said to be in similar directions when their lines of action are parallel, and they tend to produce motion towards the same quarter (fig. 3).

Fig. 3.



Forces are in dissimilar directions when their lines of action are parallel, but they tend to produce motion towards opposite quarters (fig. 4).

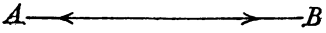
Fig. 4.



Opposite forces are forces acting in the same line, but in dissimilar directions (fig. 5).

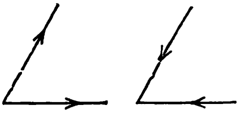
The terms "similar," and "dissimilar," are also applied to the directions of the forces which are not parallel, but which, meeting at a point, have an angle between their lines of action.

Fig. 5.



When the forces, as in fig. 6, both act from, or both towards the angle, they are said to be in *similar* directions.

Fig. 6.



When, as in fig. 7, one acts from, and the other towards the angle, they are said to be in *dissimilar* directions.

When there are four forces acting, as in fig. 8, along the sides of a parallelogram, the adjacent forces are in *similar*, the parallel forces in *dissimilar* directions.

Fig. 7.

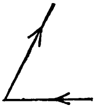
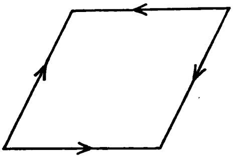
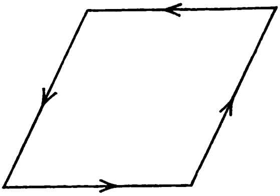


Fig. 8.



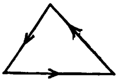
§ 24. When the four forces are as in fig. 9, the adjacent forces are in dissimilar directions, and the parallel forces are also in dissimilar directions.

Fig. 9.



In this case, and in every such case as that illustrated by fig. 10, the forces are said to be in consecutive directions round the periphery of the diagram.

Fig. 10.



§ 25. When there are several forces acting on different points, but whose lines of action all meet in a point, it is most generally convenient to take, as their common point of reference, the point of intersection of their lines of action.

When the lines of action do not all pass through one point, the points of reference may be taken in any convenient positions, as at the surface of the body on which the forces act.

# DIVISION I.

## STATICS.

§ 26. Statics has been defined (§ 3) as the science in which the balancing of forces is investigated.

§ 27. Forces can act either on a material point (§ 11), or on a rigid body (§ 12). The investigation may, therefore, be conveniently conducted under two heads:—

1°. Equilibrium of a material point.
2°. Equilibrium of a rigid body.

§ 28. Under the first head, we shall consider forces according to the following classification:—

Forces acting on a point in one line.
Forces acting on a point in different lines, but in one plane.
Forces acting on a point in lines which do not lie in one plane.

§ 29. Under the second head we shall treat of parallel forces in one plane; then of parallel forces not in one plane; then of couples; and lastly, in general, of the equilibrium of a rigid body under any circumstances whatever.

§ 30. We shall then consider friction, the mechanical powers, dynamical energy, and dynamical effect, and the principle of virtual velocities.

Before entering on the first part we shall give some definitions, axioms, and preliminary propositions.

# CHAPTER I.

## DEFINITIONS.—AXIOMS.—PRELIMINARY PROPOSITIONS.

### DEFINITIONS.

§ 31. I. *Equilibrium.*—When any number of forces act upon a body, which is either at rest or in motion, but do not alter its state whether of rest or of motion, the forces are said to be in equilibrium.

§ 32. II. *Resultant.*—When any number of forces act upon a body, and are not in equilibrium, and when there is one force capable of producing the same effect as the system of forces, this one force is called the resultant of the system.

§ 33. III. *Balancing Force.*—When any number of forces acting upon a body are not in equilibrium, but are capable of being balanced by a single force, this is called their balancing force.

Corollary.—A force, equal and opposite to any single force of a mutually balancing system of forces, is the resultant of all the others.

§ 34. IV. *Component.*—A component force is one force of a system. A component of a force is a single force which, with others, has that force as resultant. A component of a system is either one of the forces of that system, or is a single force which, with others, constitutes a system equivalent to that system.

§ 35. V. *Couple.*—A couple is a system of two equal parallel forces, in dissimilar directions. (Poinsot.)

§ 36. VI. *Arm of a Couple.*—The perpendicular distance between the lines of the forces.

§ 37. VII. *Moment* of any physical agency is the numerical measure of its importance. Thus, the moment of a force round a point or round a line, signifies the measure of its importance as regards producing or balancing rotation round that point or round that line.

§ 38. VIII. *Moment of a force round a point.*—The product of the magnitude of the force into the shortest distance of its line from that point.

§ 39. IX. *Moment of a Couple.*—The product of the magnitude of either force into the arm of the couple.

§ 40. X. *Axis of a Couple.* A line drawn from any chosen point of reference perpendicular to the plane of the couple, of such magnitude and in such direction as to represent the magnitude of the moment, and to indicate the direction in which the couple tends to turn.

The most convenient rule for fulfilling the latter condition is this:—Hold a watch with its centre at the point of reference and with its plane parallel to the plane of the couple. Then, according as the motion of the hands is contrary to or along with the direction in which the couple tends to turn, draw the axis of the couple through the face or through the back of the watch.

§ 41. XI. *Effective Component.*—If, when a force has been resolved into two components, one of them is found to be balanced, or in any way rendered inoperative, the other which is left unbalanced and free to act, is called the effective component.

§ 42. XII.. Effective component of a force round a line or axis. Resolve the force into two rectangular components, at any point in its line of action, one parallel to the axis. The other is called the effective component, relatively to rotation round that line or axis.

Geometrical remark.—The plane of this resolution is the plane through the line of the force, and parallel to the axis. The effective component is along the line in which the plane of resolution cuts a plane perpendicular to the axis through the chosen point in the line of the force. The direction of the effective component is perpendicular to the direction of the axis, although these two lines do not meet.

§ 43. XIII. The moment of a force, round any line or axis, is the product of its effective component round that line, into its shortest distance.

Geometrical remark.—The shortest distance of the effective component is the same as the shortest distance of the force itself from the axis.

§ 44. XIV. *Central Axis.*—The central axis is the line in which a force can be found, such, that with the smallest possible couple it can produce the effect of a given system of forces.

## AXIOMS.

§ 45. I. Principle of the superposition of forces in equilibrium.—If a material point or rigid body be in equilibrium, when acted on by a system of forces, and again, if it be in equilibrium when acted on by another system of forces, it will be in equilibrium when acted on by the two systems simultaneously.

§ 46. II. Principle of the transmission of force.—A rigid body, acted on by two equal directly opposed forces applied at different points, and by no other forces, is in equilibrium.

§ 47. III. Two forces not equal and directly opposed, cannot be in equilibrium with one another.

§ 48. IV. Any two forces, in lines through one point, may be balanced (if not already in equilibrium) by a single force in a line through that point.

Corollary 1. Any two forces acting on a material point have a resultant.

Cor. 2. Any number of forces acting on a material point, and not in equilibrium, have a resultant.

Remark.—If a system of forces acting on a material point is in equilibrium, the forces may be said to have a resultant which is equal to zero.

§ 49. V. If any system of forces has a resultant, any other equal and similar system of forces has a resultant which is equal and similarly situated.

§ 50. VI. The line of the resultant of any two forces acting on a material point is contained in the angle between their lines of action.

§ 51. VII. The line of the resultant of two equal forces acting upon a material point, bisects the angle between their lines of action.

§ 52. VIII. "To every action there is an equal and opposite reaction, or, the mutual actions between two bodies are equal and directed oppositely." (Newton.)

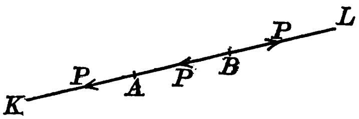
## PRELIMINARY PROPOSITIONS.

§ 53. I. Any number of forces on a material point or rigid body, which have a resultant, may be removed and their resultant applied instead of them, without altering their effect, whether for equilibrium or motion.

By hypothesis the system of forces has a resultant. Introduce a pair of balancing forces, namely, a force equal to the resultant, and one equal and opposite to it. This new system is (Axiom I) equivalent to the given system. Let all the forces be divided into two systems,—a group of forces, and a single force, thus:—The original forces and the force equal and opposite to their resultant on the one hand, the resultant of the given forces on the other. But the given forces and a force equal and opposite to their resultant balance, (§ 33): and the resultant is left alone acting on the body. In this way a system of forces may be removed, and their resultant substitued for them.

§ 54. II. A force may be transmitted to any point in the line of its action, without altering its effect on a rigid body.

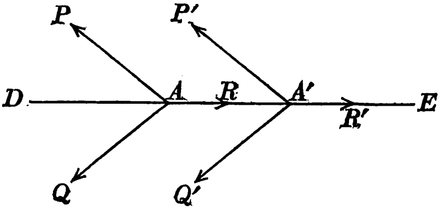
Fig. 11.



Let P (fig. 11), be a force acting on B in the line BA. Superimpose the equilibrating system, P on A in AK, in the same line as AB, and P on B in BL in the opposite direction, and each equal to P in AB. These two forces being (Axiom II) in equilibrium, will not affect the condition of the body (Axiom I). But the forces P on B, in BA, and P on B, in BL, are in equilibrium, for they are equal and act oppositely in the same straight line. They may therefore be removed, and there is then left P on A, in AK, equivalent to the original force P on B, in BA; that is, the force P has been transmitted to AK in the line of its action.

§ 55. III. Two forces may be transferred to parallel lines, through any point in the line of their resultant, without altering their effect on a rigid body.

Fig. 12.

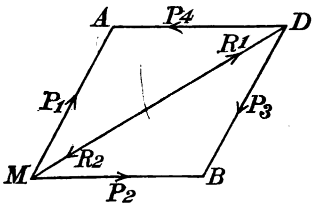


Let P and Q (fig. 12), be the original forces acting on A, and having their resultant in the line DE. Then, if P′ and Q′ be forces equal and parallel to P and Q, and act on another point, A′, in DE, their resultant must be in DE also, the circumstances of the two systems being precisely similar (Axiom V). Let R′ be a force equal and opposite to their resultant. The three forces, P′, Q′, R′, will then be in equilibrium, and they may be superimposed upon the body without affecting its state (Axiom I). Transmit (Prop. II, § 54) the force R′ acting on A′, to R acting on A. The system, P, Q, R, may then be removed, and the forces P′ and Q′ are left acting equivalently to the original forces P and Q.

§ 56. Particular case.—A very important case of this proposition is the following:—Two equal forces may be transferred to parallel lines through any point. For, since their forces are equal, their resultant (Axiom VII) will be in the line bisecting the angle between them. The following is a geometrical form of this proposition:—Forces represented by two conterminous sides of a rhombus, may be transposed to similar directions in the opposite sides without altering their effect on a rigid body. For, either diagonal of a rhombus bisects the angles through which it is drawn.

§ 57. V. Four equal forces, represented by the four sides of a rhombus, balance one another, if each pair which are not parallel are in similar directions relatively to the angle between them, and each parallel pair are in dissimilar directions.

Fig. 13.



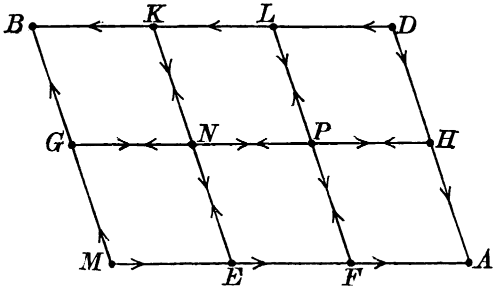
Let $P_1$, $P_2$ (fig. 13) be one pair, and $P_3$, $P_4$ another pair of four forces, which are represented by the four sides of the rhombus, A B, and of which, $P_1$, $P_2$, act from M towards A and B, and $P_3$, $P_4$, from D towards A and B. Any one of the pairs not parallel, has a resultant along the diagonal of the rhombus, through the point in which their lines meet, and which is equal and opposite to the resultant of the remaining pair. Hence, the four are in equilibrium.

§ 58. VI. Four forces, represented by the four sides of a parallelogram, and acting in similar directions, relatively to the angles, and dissimilar directions in the parallels, balance one another if its adjacent sides are commensurable.

Let M A D B (fig. 14) be a parallelogram, and let M A and MB, adjacent sides, be commensurable. Then, the forces represented by MA, MB, DB, DA, balance, if their directions are from M to A, from M to B, from D to A, and from D to B respectively.

Divide M A and M B into parts, M E, E F, F A, and M G, G B, each equal to their greatest common measure. Through

Fig. 14.



the points of division, E, F, draw E K, F L parallel to M B or A D; and through G draw G H parallel to M A or B D. Suppose the number of parts into which the force M A is divided to be three. Transmit two of them to E and F respectively; and similarly supposing M B to contain two of the equal parts, transmit one of them to G. In like manner divide D B and D A, and transmit the parts respectively to L and K, and to H. Let each of these equal forces be denoted by $f$.

Introduce in E N a pair of equilibrating forces, each equal to $f$, one acting from N, and one from E; in F P, a similar pair, the one acting from P, the other from F; in N K a pair, the one acting from K, the other from N; and so in P L: also, a pair in G N, the one towards G, the other from it; and so in N P and P H. Instead of the original system there are now twenty-four equal forces. These may be viewed as six separately equilibrating groups of four forces each. For, in the sides of each of the rhombuses, M N, E P, F H, &c., there are four equal forces acting in similar directions relatively to its angles, which, therefore, are in equilibrium. (Prop. V, § 57). Thus we see that the whole system is in equilibrium, and therefore the original system, to which it is equivalent, must be in equilibrium.

Remark.—This proposition being proved for any commensurable forces, may be held as proved, with perfect rigour, for incommensurable forces: and there is not, in strictness, any necessity for a formal and distinct demonstration of this case. If two forces have a common measure at all, the proposition applies to them expressly. But if the numbers expressing the magnitudes of the forces be absolutely incommensurable, they may be approximately, but cannot be rigorously, expressed in terms of a common measure. Since, however, the approximation can be made as close as we please, we may, by regarding the case of two absolutely incommensurable forces as an extreme or limiting case of the general proposition, apply the above demonstration, in the limit, to them also. A distinct proof, however, will be given for the case of incommensurable forces, as affording another and a useful mode of viewing the extension, rather than as essential to the rigour of our course.

# CHAPTER II.

## EQUILIBRIUM OF FORCES ACTING IN ONE LINE.

§ 59. The problem which has to be solved under the first head of Statics (§ 28) is—" To find the resultant of any number of forces acting in any direction whatever upon a material point," in which is included, "the determination of the conditions of equilibrium; or, as we may otherwise view the question, to find under what conditions the resultant is equal to zero."

§ 60. For convenience of solution we shall follow the order already given (§ 29); and we shall, therefore, now consider a set of forces acting all in one line; and first, the case of all in the same direction; then secondly, some in one direction and some in the opposite.

§ 61. I. To find the resultant of any number of forces acting on a material point in one line and in the same direction.

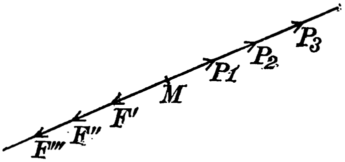
The resultant of all the forces is equal to their sum. The effect of a number of forces, all conspiring, is just the same as if one force, equal to the sum of the different separate forces, were acting instead of them.

§ 62. II. To find the resultant of any number of forces acting on a material point in one line, but not in the same direction.

According to § 61 all the forces in one direction are equivalent to one force, which is their sum; and similarly the forces in the other direction. These two forces, being opposed, may balance or not. If they do balance they are equal, and the resultant of the whole system is therefore zero. If they do not there will be a resultant in the direction of the force which preponderates, and it will be equal to the excess of the greater above the less; or, the resultant will be expressed numerically by the difference between the two opposing partial resultants. It is obvious that we may here use, with advantage, the rule for performing an algebraic summation, which is:— Add together all the quantities which have the positive sign; and add separately all those which have the negative sign. Subtract the greater of these sums from the less, and if the balance is on the side of those which have the positive sign, prefix it to the balance; if on the side of those with the negative sign, prefix the sign minus, the sum being in this case a negative quantity.

§ 63. To apply this to our case of forces, attach arbitrarily the positive sign to those acting in one direction, and the negative sign to those in the other.

Fig. 15.



Let $P_1$, $P_2$, $P_3$, (fig. 15) be considered positive, Then the other set, of which the absolute magnitudes and directions are indicated in the diagram by $F'$, $F''$, $F'''$, and the corresponding arrow-heads, will be the negative. Add, algebraically, the forces, and we get

$$P_1 + P_2 + P_3 - F' - F'' - F'''$$

as the value of the resultant, its direction being determined by the sign to be prefixed to the numerical result when the operation is concluded.

§ 64. The general conclusion is as follows:—If any number of forces act on a material point in the same line, these forces are equivalent to a single resultant, which is equal to their algebraic sum.

If the algebraic sum is zero, the forces are in equilibrium.

# CHAPTER III.

## COMPOSITION OF FORCES ALONG DIFFERENT LINES IN ONE PLANE.

§ 65. The case that has next to be considered is that of forces, which act on a point in different lines all lying in one plane (§ 28). The simplest instance is that of two forces in different lines. Their resultant is found by the "parallelogram of forces,"—a proposition, which is the foundation of the whole of Statics.

§ 66. Parallelogram of forces.—

The resultant of two forces, acting on a material point, is represented in direction and magnitude by the diagonal, through that point, of the parallelogram described upon lines representing the forces.

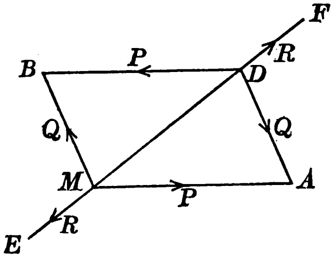
§ 67. This proposition asserts, 1° that the resultant of two forces acting on a point is along the diagonal; 2° that it is represented in magnitude by the length of that diagonal.

The demonstration falls, therefore, naturally into two parts. The first of these may, for convenience, be subdivided into two cases:—

*a*, When the forces are commensurable.
*b*, When the forces are incommensurable.

§ 68. Part I. Case *a*. The resultant of two commensurable forces, acting on a material point, is along the diagonal of the parallelogram described upon lines representing the forces.

Fig. 16.



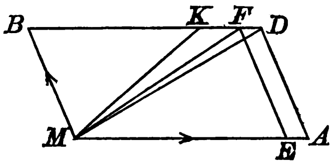
Let M (fig. 16) be a point acted on by the forces P in M A, and Q in M B, which are respectively represented by those lines (§ 22). On them construct a parallelogram, M A D B, and draw the diagonal, M D. The resultant of the given forces lies along M D.

Through M draw M E

opposite to the unknown direction of the resultant of the given forces, and in it let a force, R, equal to that resultant, act. This will balance the two given forces. Introduce (Axiom I) at D an equilibrating system of forces, namely, a force equal to P along D B; a force equal to Q along D A; and a force, R in DF, equal and opposite to their resultant. Instead of the given forces there is now a system of six forces in equilibrium. By preliminary proposition VI (§ 58), P in M A, Q in M B, P in D B, Q in D A, are in equilibrium. They may, therefore, be removed, leaving two forces which must balance each other, R in ME and R in DF. These two forces must act (Axiom III) in one and the same straight line M D. But R in M E is opposite to the resultant of the given forces, therefore, the resultant of the given forces is along M D.

§ 69. Part I. Case *b*. The preceding demonstration, like that of preliminary proposition VI (§ 58), includes the case of incommensurable forces. A separate demonstration, however, may be given for this case, as follows:—

Fig. 17.



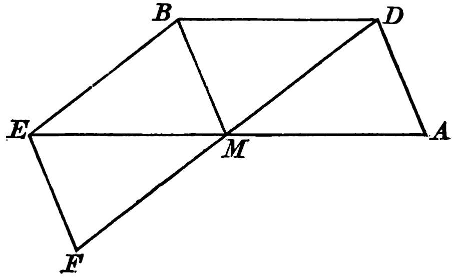
Let M A and M B (fig. 17) represent two incommensurable forces acting on M. On these lines construct the parallelogram B A. The resultant of the forces lies along M D, the diagonal.

For, if it does not, let it lie in some other direction, say along M K. Divide M B into any number of equal parts, each smaller than K D, and measure off along M A as many of them as possible. Since M A and M B are incommensurable, there will be something less than one of these parts remaining. Let E A be this remainder, and through E draw E F, parallel to M B or A D. Let the force A M be divided into two parts, represented respectively by M E and E A, but both acting at M. Then, M B, M E, act on M, and as they are commensurable, their resultant must lie along M F, the diagonal of the parallelogram, M B F E (§ 68). Hence, a certain force along M F, and the force E A, are equivalent to the given forces; and their resultant must lie (Axiom VI) in the angle between them, namely, in the angle F M A. Hence, it cannot be along M K. In like manner it can be shown that in no other line than M D, in the angle B M A, can the resultant of the two forces possibly lie. M D is therefore (Ax. VI) the direction of the resultant.

§ 70. Part II. The resultant of two forces, acting on a material point, is represented in magnitude by the diagonal, through that point, of the parallelogram, described upon lines representing the forces.

Let M (fig. 18) be a material point, and let M A, M B, represent two forces acting upon it. Through A draw a line A D, equal and parallel to M B, and through B a line B D, equal and parallel to M A, and let them meet at D. Join M D. M D represents the resultant of the given forces in magnitude. Produce D M, through M, to any distance, F, so that M F shall represent the required magnitude of the resultant. Let a force represented by M F be applied at M, which will balance the given forces, being equal and (§ § 68, 69) opposite to their resultant. Again, the resultant of M F and M B must (§ § 68, 69) be in the direction of the diagonal of the parallelogram described on their lines. Let that parallelogram be F B. Join M E. The force in M A balances M F and M B, and the resultant of the same forces is in M E. Therefore, the force in M A, and the force in M E, balance: but two forces cannot balance unless directly opposed (Axiom III); therefore, M A and M E are directly opposed, that is, M E acts in the same straight line as M A; E M A, therefore, is a straight line. Since E M A is a straight line, and M A is parallel to B D, E M is parallel to BD. And since, by construction, E B is parallel to M F, which is a part of the same straight line D M produced, E B is parallel to

Fig. 18.

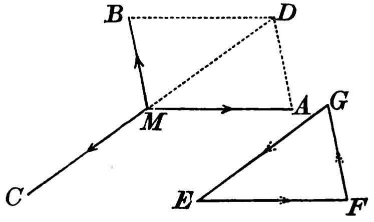


DM. Therefore, E B D M is a parallelogram, and the sides, E B, D M, are therefore equal. Again, B F is a parallelogram, and F M is therefore equal to E B. But E B is equal to M D, wherefore, also, F M is equal to M D. But F M represents, in magnitude, a force equal and opposite to the resultant of the given forces. Therefore D M equal to it, and on the other side of M, represents in magnitude that resultant, which was to be shewn.

§ 71. Parallelogram of forces stated symmetrically as to the three forces concerned. If the lines representing three forces acting on a material point be equal and parallel to the sides of a triangle, and in directions similar to those of the three sides when taken in order round the triangle, the three forces are in equilibrium.

Let G E F (fig. 19) be a triangle, and let M A, M B, M.C, be respectively equal and parallel to the three sides, E F, F G, G E of this triangle, and in directions similar to the consecutive directions of these sides in order. The point M is in equilibrium.

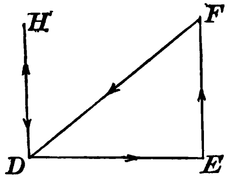
Fig. 19



For if on any two of the three lines representing the forces acting on M, we describe a parallelogram, its diagonal through M may be proved to be equal to the third, and in the same line with it. If, for example, we draw A D equal and parallel to MB, and BD parallel to MA, meeting AD in D, and join DM; we have M A, A D, and the contained angle, A, respectively equal to E F, F G, and the contained angle, F. Hence, of the triangles M A D, E F G, the third sides, M D, E G, are equal, and the remaining angles, A D M, D M A, are respectively equal to F G E, G E F. But E G is equal to C M, by hypothesis, therefore C M is equal to M D. And since M A is parallel to E F, and the angles, D M A, G E F have been proved to be equal, the lines, DM, G E are parallel. But C M is parallel to G E. Hence, C M and M D lie in one straight line. We conclude that M C is equal and opposite to the resultant of M A and M B; that is to say (§ 33), the three forces are in equilibrium.

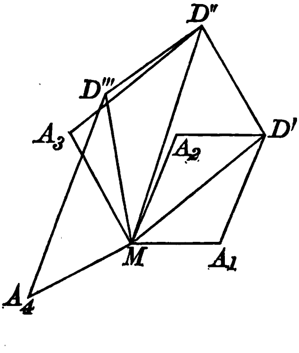
§ 72. Triangle of Forces. Let three forces act in consecutive directions round a triangle, D E F (fig. 20), and be represented respectively by its sides: they are not in equilibrium, but are equivalent to a couple (Def. V, § 35). To prove this, through D draw D H, equal and parallel to E F, and in it introduce (Ax. I) a pair of balancing forces, each equal to E F. Of the five forces, three, D E, D H and F D are in equilibrium, and may be removed; and there are then left two forces, E F and H D, equal, parallel, and in dissimilar directions, which constitute a couple.

Fig. 20.



§ 73. To find the resultant of any number of forces in lines through one point.

Fig. 21.



Let M $A_1$, M $A_2$, M $A_3$, M $A_4$, (fig. 21), represent four forces acting on M, in one plane; required their resultant.

Find by the parallelogram of forces, the resultant of two of the forces, M $A_1$ and M $A_2$. It will be represented by M D′. Then similarly, find M D″, the resultant of M D′ (the first subsidiary resultant), and M $A_3$, the third force. Lastly, find M D‴, the resultant of M D″ and M $A_4$. M D‴ represents the resultant of the given forces.

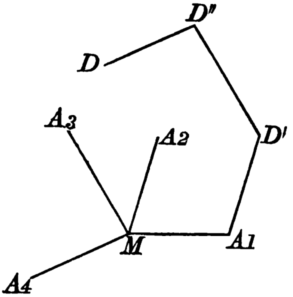
Thus, by successive applications of the fundamental proposition, the resultant of any number of forces in lines through one point can be found.

Remark.—The forces thus compounded are not necessarily in one plane.

§ 74. In executing this construction, it is not necessary to describe the successive parallelograms, or even to draw their diagonals. It is enough to draw through the given point a line equal and parallel to the representative of any one of the forces;

through the point thus arrived at, to draw a line equal and parallel to the representative of another of the forces, and so on till all the forces have been taken into account. In this way we get such a diagram as the annexed (Fig. 22).

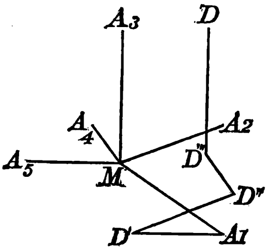
Fig. 22.



The several given forces may be taken in any order, in the construction just described. The resultant arrived at is necessarily the same, whatever be the order in which we choose to take them, as we may easily verify by elementary geometry.

In Fig. 23, the order is M $A_1$, M $A_5$, M $A_2$, M $A_4$, M $A_3$.

Fig. 23.



§ 75. If, by drawing lines equal and parallel to the representatives of the forces, a closed figure is got, that is, if the line last drawn leads us back to the point from which we started, the forces are in equilibrium. If, on the other hand, the figure is not closed (fig. 22 or 23), the resultant is obtained by drawing a line from the starting point to the point finally reached; (from M to D): and a force represented by M D will equilibrate the system.

§ 76. Hence, in general, a set of forces represented by lines equal and parallel to the sides of a complete polygon, are in equilibrium, provided they act on one point, in directions similar to the directions followed in going round the polygon in one way.

§ 77. Polygon of forces. The construction we have just considered, is sometimes called the polygon of forces; but the true polygon of forces, as we shall call it, is something quite different. In it the forces are actually along the sides of a polygon, and represented by them in magnitude. Such a system must clearly have a turning tendency, and it may be demonstrated to be reducible to one couple (see below, Chap. VIII).

# CHAPTER IV.

## FORCES IN ONE PLANE.—APPLICATION OF TRIGONOMETRY.—ELEMENTARY FORMULÆ.—CASES.—DIRECT AND INVERSE PROBLEMS.—COMPOSITION.—RECTANGULAR RESOLUTION.—RESULTANT.

§ 78. In Chapter III has been shewn the principle involved in finding the resultant of any number of forces. We have now to find a method, more easy than the parallelogram of forces affords, for working it out in actual cases, and especially for obtaining a convenient specification of the resultant. The instrument employed for this purpose is Trigonometry.

§ 79. A distinction may first be pointed out between two classes of problems, direct and inverse. Direct problems are those in which the resultant of forces is to be found; inverse, those in which components of a force are to be found. The former class is fixed and determinate; the latter is quite indefinite, without limitations not yet stated. A system of forces can produce only one effect; but an infinite number of systems can be obtained, which shall produce the same effect as one force. The problem, therefore, of finding components must be, in some way or other, limited. This may be done by giving the lines along which the components are to act. To find the components of a given force in any three given directions, is, in general, as we shall see, a perfectly determinate problem.

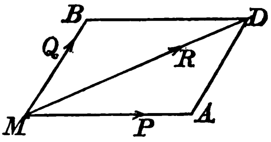
Finding resultants is called Composition of Forces.

Finding components is called Resolution of Forces.

§ 80. Composition of Forces.

Required in position and magnitude the resultant of two given forces acting in given lines on a material point.

Fig. 24.



Let M A, M B (fig. 24) represent two forces, P and Q, acting on a material point M. Let the angle B M A be denoted by I. Required the magnitude of the resultant, and its inclination to the line of either force.

Let R denote the magnitude of the resultant; let $\alpha$ denote the angle D M A, at which its line M D is inclined to M A, the line of the first force P; and let $\beta$ denote the angle D M B, at which it is inclined to M B, the direction of the force Q.

Given P, Q, and I: required R, and $\alpha$ or $\beta$.

We have

$$MD^2 = MA^2 + MB^2 - 2\,MA.\,AD \times \cos MAD.$$

Hence, according to our present notation,

$$R^2 = P^2 + Q^2 - 2\,P\,Q \cos (180^\circ - I),$$

or

$$R^2 = P^2 + Q^2 + 2\,P\,Q \cos I.$$

Hence $$R = (P^2 + Q^2 + 2\,P\,Q \cos I)^{\frac{1}{2}} \quad . \quad . \quad . \qquad (1).$$

To determine $\alpha$ and $\beta$ after the resultant has been found; we have

$$\sin DMA = \frac{MB}{MD} \sin MAD,$$

or

$$\sin \alpha = \frac{Q}{R} \sin I \quad . \quad . \quad . \quad . \quad . \qquad (2),$$

and, similarly,

$$\sin \beta = \frac{P}{R} \sin I \quad . \quad . \quad . \quad . \quad . \qquad (3).$$

§ 81. These formulæ are useful for many applications; but they have the inconvenience that there may be ambiguity as to the angle, whether it is to be acute or obtuse, which is to be taken when either sin $\alpha$ or sin $\beta$ has been calculated. If I is acute, both $\alpha$ and $\beta$ are acute, and there is no ambiguity. If I is obtuse, one of the two angles, $\alpha$, $\beta$, might be either acute or obtuse; but as they cannot be both obtuse, the smaller of the two must, necessarily, be acute. If, therefore, we take the formula for sin $\alpha$, or for sin $\beta$, according as the force P, or the force Q is the greater, we do away with all ambiguity, and have merely to take the value of the angle shown in the table of sines. And by subtracting the value thus found, from the given value of I, we find the value, whether acute or obtuse, of the other of the two angles, $\alpha$, $\beta$.

§ 82. To determine $\alpha$ and $\beta$ otherwise. After the magnitude of the resultant has been found, we know the three sides, M A, A D, M D, of the triangle D M A, then we have

$$\cos DMA = \frac{MD^2 + MA^2 - AD^2}{2\,MD.\,MA},$$

or

$$\cos\alpha = \frac{R^2 + P^2 - Q^2}{2\,RP} \quad \ldots\ldots \quad (4),$$

and similarly,

$$\cos\beta = \frac{R^2 + Q^2 - P^2}{2\,RQ} \quad \ldots\ldots \quad (5),$$

by successive applications of the elementary trigonometrical formula used above for finding M D. Again, using this last-mentioned formula for $MD^2$ or $R^2$ in the numerators of (4) and (5), and reducing, we have

$$\cos\alpha = \frac{P + Q\cos I}{R} \quad \ldots\ldots\ldots \quad (6),$$

$$\cos\beta = \frac{Q + P\cos I}{R} \quad \ldots\ldots\ldots \quad (7);$$

formulæ which are convenient in many cases. There is no ambiguity in the determination of either $\alpha$ or $\beta$ by any of the four equations (4), (5), (6), (7).

Remark.—Either sign (+or−), might be given to the radical in (1), and the true line of action and the direction of the force in it, would be determined without ambiguity, by substituting in (2) and (3) the value of R with either sign prefixed. Since, however, there can be no doubt as to the direction of the force indicated, it will be generally convenient to give the positive sign to the value of R; while only in special cases, the negative sign, which with the proper interpetration of the formulæ, will lead to the same result as the positive, shall be employed.

§ 83. Another method of treating the general problem, which is useful in many cases, is this;

Let

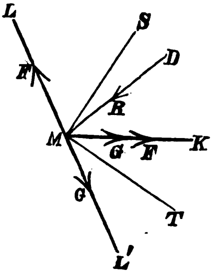
$$\tfrac{1}{2}(P + Q) = F,$$
$$\tfrac{1}{2}(P - Q) = G,$$

which implies that

$$P = F + G,$$
$$Q = F - G.$$

F and G will be both positive if the given forces are both positive, and if $P > Q$. Hence, instead of the two given forces, P and Q, we may suppose that we have on the point M (fig. 25), four forces;—two, each equal to F, acting in the same directions, M K, M L, as the given forces, and two others, each equal to G, of which one acts in the same direction, M K, as P, and the other in M L, the direction opposite to Q. Now, according to Axiom VII (§ 51), the resultant of the two equal forces, F, bisects the angle between them, K M L; and by the investigation of Case I, (§ 84) below, its magnitude is found to be $2\,F \cos \tfrac{1}{2}\,I$. Again, the resultant of the two equal forces, G, is similarly seen to bisect the angle, K M L$'$, between the line of the given force, P, and the continuation through M of the line of the given force, Q; and to be equal to $2\,G \sin \tfrac{1}{2}\,I$, since the angle K M L$'$ is the supplement of I. Thus, instead of the two given forces in lines inclined to one another at the angle I, which may be either an acute, or an obtuse, or a right angle, we have two forces, $2\,F \cos \tfrac{1}{2}\,I$ and $2\,F \sin \tfrac{1}{2}\,I$, acting in lines, M S, M T, which bisect the angles L M K and K M L$'$, and therefore are at right angles to one another. Now, according to Case VI, (§ 89) below, we find the resultant* of these two forces by means of the following formulæ :—

Fig. 25.



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* In the diagram the direction of the balancing force is shown by the arrow-head in the line D M.

$$\tan SMD = \frac{2\ G \sin \frac{1}{2}\ I}{2\ F \cos \frac{1}{2}\ I},$$

and

$$R = 2\ F \cos \tfrac{1}{2}\ I \sec S\,M\,D,$$

or

$$\tan(\tfrac{1}{2}I - \alpha) = \frac{P - Q}{P + Q} \tan \tfrac{1}{2}\ I \quad . \quad . \quad . \quad . \quad (8)$$

and

$$\begin{aligned} R &= (P + Q) \cos \tfrac{1}{2}\ I \sec (\tfrac{1}{2}\ I - \alpha) \\ &= (P + Q) \cos \tfrac{1}{2}\ (\alpha + \beta) \cos \tfrac{1}{2}\ (\alpha - \beta) \quad . \quad (9) \end{aligned}$$

These formulæ might have been derived from the standard formulæ (Thomson's Trigonometry, No. 54) for the solution of a plane triangle when two sides (P and Q), and the contained angle ($\pi - I$) are given.

§ 84. We shall now investigate some cases of the general formulæ.

Case I. Let the forces be equal, that is, let $Q = P$ in the preceding formulæ.

Then, by (1), $R^2 = 2\ P^2 + 2\ P^2 \cos I = 2\ P^2\ (1 + \cos I)$.

$$= 4\ P^2 \cos^2 \tfrac{1}{2}\ I.$$

Hence $$R = 2\ P \cos \tfrac{1}{2}\ I,$$

an important result which might, of course, have been obtained directly from the proper geometrical construction in this case. Also, by (2),

$$\sin \alpha = \frac{Q \sin I}{R} = \frac{Q \sin I}{2\ Q \cos \frac{1}{2}\ I} = \sin \tfrac{1}{2}\ I,$$

which agrees with what we know by § 51 (Axiom VII), that is to say, that $\alpha = \beta = \frac{1}{2}\ I$.

§ 85. Case II. Let $P = Q$; and let $I = 120°$. Then $\cos \frac{1}{2}\ I = \cos 60° = 1$, and (§ 84) $R = P$.

The resultant, therefore, of two equal forces inclined at an angle of 120° is equal to each of them. This result is interesting, because it can be obtained very simply, and quite independently of this investigation, from a proposition which might have been deduced immediately from Axiom V (§ 49). A consideration of the symmetry of the circumstances will shew, as a consequence of the axiom, that if three equal forces be applied to a material point in lines dividing the space around it into three equal angles, they must be in equilibrium; which is perfectly equivalent to the preceding conclusion.

§ 86. Case III. Let $I = 0°$; $\cos I = 1$; then

$$R = (P^2 + Q^2 + 2\,P\,Q)^{\frac{1}{2}}$$
$$R = P + Q.$$

This is the first case of all (§ 61).

§ 87. Case IV. Let $I = 180°$; $\cos I = -1$; then

$$R = (P^2 + Q^2 - 2\,P\,Q)^{\frac{1}{2}}$$
$$R = P - Q.$$

This is also one of the elementary cases (§ 62). It is also one of the cases in which it is convenient to give sometimes the negative sign, sometimes the positive to the expression for the resultant force: for if Q be greater than P, the preceding expression will be negative, and the interpretation will be found by considering that the force which vanishes when $P = Q$, is in the direction of P when P is the greater, and in the contrary direction, or in that of Q, when P is the less of the two forces.

§ 88. Case V. Forces nearly conspiring. Let the angle I be very small, then

$$\sin I =| I;^{*} \cos I =| 1.$$

The general expressions (§ 80) therefore become,

$$R =| P + Q,$$

$$\sin \alpha =| \frac{Q\,I}{P + Q},$$

$$\sin \beta =| \frac{P\,I}{P + Q}.$$

To the same degree of approximation

$$\alpha =| \frac{Q\,I}{P + Q} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (10)$$

---

[*] The sign $=|$ is used to denote approximate equality, and the sign $=||$ will be sometimes used to denote a higher degree of approximation.

$$\beta =| \frac{P\,I}{P + Q} \quad - - - - - - - \quad (11).$$

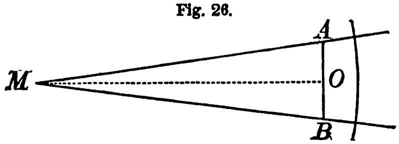
Hence $\alpha + \beta =| \frac{Q\,I + P\,I}{P + Q} =| \frac{(Q + P)\,I}{(Q + P)} = I.$

This shows that the errors in the values of $\alpha$ and $\beta$ obtained approximately by this method, compensate; one being as much above, as the other is below the true value.

We therefore conclude that the resultant of two forces very nearly conspiring is approximately equal to their sum, and approximately divides the angle between them into parts inversely as the forces.

When the angle between the forces is infinitely small, they may either conspire in acting on one point in one line; or they may act on different points in parallel lines. In either case the resultant is precisely equal to their sum. Actually conspiring forces we have already considered (§ 84, Case I); parallel forces we shall consider more particularly when we treat of the equilibrium of a rigid body. We may briefly examine the case here however. Suppose the actual points of application of the forces to be A and B (fig. 26), but let their lines meet in a point M; join A B, and let M A B be an isosceles triangle. Let this point M be removed gradually to an infinite distance in the direction of a perpendicular, M O, bisecting the line A B. The resultant will still divide the angle inversely as the forces: and as the circular measure of the angle is any arc described from M as centre divided by the radius, every such arc will be divided in the same proportion. Now, if M be infinitely distant, that is if the lines of the forces be parallel, the arc will become a straight line, and will be divided into parts inversely as the forces.

Fig. 26.



In actual cases of forces acting on a point, and very nearly conspiring, the following approximate equations show how nearly the resultant approaches the sum of the forces:—

$$\sin\theta =| \theta;\ \cos\theta =| 1;\ \cos\theta =|| 1 - \tfrac{1}{2}\theta^2.$$
$$R^2 =|| P^2 + Q^2 + 2\,P\,Q - P\,Q\,I^2$$

$$R =\| \sqrt{\left\{(P+Q)^2\left[1-\frac{P\,Q\,I^2}{(P+Q)^2}\right]\right\}}$$

$$R =\| (P+Q)\left[1-\tfrac{1}{2}\frac{P\,Q\,I^2}{(P+Q)^2}\right]$$

$$R =\| (P+Q)-\tfrac{1}{2}\frac{P\,Q}{P+Q}I^2 \;.\;\;.\;\;.\;\;. \qquad (12)$$

that is: the resultant of two forces very nearly conspiring falls short of their sum by the square of the angle between them multiplied into a quarter of their harmonic mean.*

§ 89. Case VI. Forces nearly opposed.

1°. Let the angle I be very obtuse, and the two forces exactly equal.

Let $I = \pi - \theta$, where $\theta$ is very small,

then $\frac{1}{2} I = \frac{1}{2}\pi - \frac{1}{2}\theta$.

$$\cos \tfrac{1}{2} I = \sin \tfrac{1}{2}\theta.$$

$$R = 2\,P \sin \tfrac{1}{2}\theta.$$

and since the sine of a very small angle is equal to the angle, in circular measure

$$R =| P\,\theta.$$

Hence the resultant of two equal very nearly opposed forces is proportional to the defalcation from direct opposition: being approximately equal to either of the forces multiplied into the supplement of the angle between them.

2°. If the forces are neither equal nor nearly equal, the resultant will be approximately equal to their difference.

We have as before,

$$\cos I =| -1,$$

$$R^2 =| P^2 + Q^2 - 2\,P\,Q.$$

Therefore $R =| P - Q.$

---

* The Harmonic Mean of two numbers is the reciprocal of the mean of their reciprocals. Thus the harmonic mean of P and Q is $\frac{P\,Q}{P+Q}$.

$$\sin\alpha =| \frac{Q(\pi - I)}{P - Q}$$

$$\sin\beta =| \frac{P(\pi - I)}{P - Q}.$$

The ambiguity as to whether the acute angle, shown in the table, or its supplement, is to be chosen in either case, may be removed by considering which of the two forces is the greater.

Thus, as we suppose P to be greater than Q, $\alpha$ is acute, and therefore

$$\sin\alpha =| \alpha =| \frac{Q(\pi - I)}{P - Q}$$

and $\beta$ is obtuse.

Therefore,

$$\beta =| \pi - \frac{P(\pi - I)}{P - Q}$$

or

$$\beta =| \frac{PI - Q\pi}{P - Q}.$$

We find, by addition,

$$\alpha + \beta = \frac{P - Q}{P - Q} I = I,$$

and conclude, as in the former case, that the errors in the approximate values of $\alpha$ and $\beta$ compensate, one being as much above, as the other is below, the true value.

It is only when R is comparable in magnitude with P and Q, that the foregoing solution is applicable.

But if P exceeds Q, or if Q exceeds P, by any difference which is considerable in comparison with either, the formulæ hold.

Let us suppose now that, while P remains of any constant magnitude, Q is made to increase from nothing, gradually, until it becomes first equal to, and then greater than P, the angle I remaining constant. The angle $\alpha$ will increase very slowly, according to the approximate formula (10), until Q becomes nearly equal to P. Then as the value of Q is increased until it becomes greater than P, the value of $\alpha$ will increase very rapidly through nearly two right angles, until it falls but little short of I, when its supplement will be approximately expressed by the formula (10).

In this transition, from $Q < P$ to $Q > P$, the direction and magnitude of the resultant are most conveniently found by means of (§ 83) the last of the three general methods given above for determining the resultant of two forces.

Thus, instead of the two given forces we may substitute two forces in lines bisecting respectively the obtuse angle L M K, or I, and the acute angle K M L′ (fig. 25), and of magnitudes which approximate to $\frac{1}{2}(P+Q)(\pi-I)$, and $P-Q$ respectively, when I is nearly two right angles.

We infer, finally, that, however nearly P and Q are equal to one another, the approximate formulæ of § 89, 2° hold, provided only $\frac{1}{2}(P+Q)(\pi-I)$ is a small fraction of $P-Q$.

§ 90. Case VII. Let $I = 90°$; $\cos I = 0$, $\sin I = 1$;

then
$$R = (P^2 + Q^2)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13),$$

and
$$\left.\begin{aligned} \sin\alpha &= \frac{Q}{R} \\ \sin\beta &= \frac{P}{R} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

In this case, $\beta$ being the complement of $\alpha$, $\sin\beta = \cos\alpha$;

Hence
$$\cos\alpha = \frac{P}{R}.$$

Lastly, since
$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}.$$

we deduce
$$\tan\alpha = \frac{Q}{P}: \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

and
$$R = P\sec\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

Remark.—These formulæ have thus been derived from the general expressions (§ 80); but they can also be very readily got from a special geometrical construction, corresponding to the case in which the lines of the forces are at right angles to one another, the principles to be used being (1) the parallelogram of forces; (2) Euclid I., XLVII.; and (3) the trigonometrical definitions of sine, cosine, and tangent.

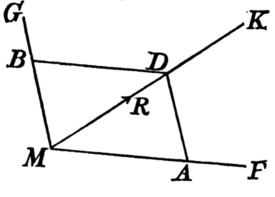
§ 91. This case is of importance, for it affords us the formulæ for rectangular resolution; by the aid of which we shall, a little later, proceed to calculate the resultant of any number of forces in one plane. We might calculate the resultant by applying the elementary formulæ (§ § 80, 81, 82), to repetitions of the parallelogram of forces. But this process would be very complicated and tedious, if the forces were numerous, and their magnitudes and angles given in numbers; and we shall see that it may be avoided by resolving all the forces along two lines at right angles to one another, and thus obtaining as equivalent to them, two forces along these lines.

We shall first consider the general inverse problem (§ 79), or the resolution of forces.

§ 92. If a force acting on a material point, and two lines in one plane with the line of that force, be given, it is possible to find determinately two forces along those lines, of which the given force is the resultant.

The two forces thus determined are called the components of the given force along the given lines, and if we substitute these two forces for the given force, we are said to resolve the given force into two forces along the given lines; or, to resolve the force along the given lines.

Fig. 27.



§ 93. Geometrical solution. Let M (fig. 27) be the given point; R, the given force acting on it in the line, M K; and M F and M G the given lines.

It is required to find the components along M F and M G of R in M K.

Take any convenient length M D to represent the magnitude of the given force, R. Through D draw D A parallel to G M, and let it cut M F in A; and also through D draw D B parallel to F M, and let it cut M G in B; M A and M B represent the required magnitudes of the components.

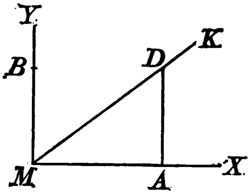
§ 94. Trigonometrical solution. If the angle K M F (fig. 27) be given $=\alpha$, and K M G $=\beta$, and if the required component of the given force R along M F be denoted by P, and the component along M G by Q, we deduce from equations (2) and (3) (§ 81), the following:—

$$P = \frac{R \sin \beta}{\sin (\alpha + \beta)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17).$$

$$Q = \frac{R \sin \alpha}{\sin (\alpha + \beta)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

§ 95. When the given lines of resolution are at right angles to one another, these expressions are modified in the manner shown above (§ 90, Case VII), or we may find them at once from the geometrical construction proper for the case, thus:—

Fig. 28.



Let M X, M Y (fig. 28) be the given lines; X M Y = 90°, and MD = R. Also as before DMA = $\alpha$, and DMB = $\beta$. Draw DA parallel to YM, or perpendicular to MX, and make M B = A D. Then in the right-angled triangle M A D, M A = M D cos D M A, and A D = M D sin D M A.

Hence, since M A represents the component along M X, and M B the component along M Y,

$$P = R \cos \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

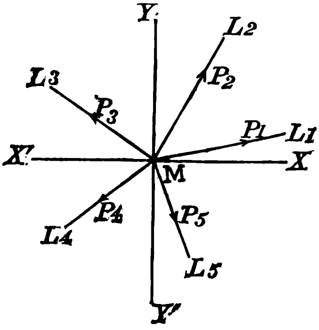
and

$$Q = R \sin \alpha, \text{ or } Q = R \cos \beta \quad . \quad . \quad . \quad . \quad (20).$$

Hence, in rectangular resolution, the component, along any line, of a given force, is equal to the product of the number expressing the given force, into the cosine of the angle at which its direction is inclined to that line.

§ 96. Application of the Resolution of Forces. Let there be a number of forces $P_1$, $P_2$, $P_3$, $P_4$, $P_5$, (fig. 29), acting respectively in lines M $L_1$, M $L_2$, M $L_3$, M $L_4$, M $L_5$, on a material point M; required their resultant.

Fig. 29.



Through M, draw at right angles to each other, and in the same plane as the given forces, two lines, XX′ and YY′, which may be called lines or axes of resolution. Let the angle which the resultant forms with the line of resolution M X, be denoted by $\theta$, and let the angles, which the lines of the forces make respectively with the lines of resolution, be denoted by $\alpha_1, \beta_1$ ; $\alpha_2, \beta_2$ ; $\alpha_3, \beta_3$ ; &c.; that is $L_1MX = \alpha_1$, $L_1MY = \beta_1$, and so on.

The angles $\beta_1$ $\beta_2$, &c., are merely the complements of $\alpha_1 \alpha_2$, &c., and, except for the sake of symmetry, they need not have been introduced into our notation.

Resolve (§ 95) the first force $P_1$, into two components, one along M X, and one along M Y. These are

$P_1 \cos \alpha_1$ along M X, which force may be denoted by $X_1$,

and $P_1 \sin \alpha_1$ along M Y, which force may be denoted by $Y_1$.

Treat all the other forces in like manner, thus reducing them to components along MX and M Y; and add together the components along each of the lines of resolution. Then if X denote the sum of the components along M X, and Y the sum of the components along M Y, we have

$$X = P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + P_3 \cos \alpha_3 + P_4 \cos \alpha_4 + P_5 \cos \alpha_5.$$

$$Y = P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + P_3 \sin \alpha_3 + P_4 \sin \alpha_4 + P_5 \sin \alpha_5.$$

Lastly, to find the resultant of X and Y.

(§ 90). $$R = \sqrt{(X^2 + Y^2)} \quad \ldots\ldots \quad (21),$$

and $$\cos \theta = \frac{X}{R} \quad \ldots\ldots\ldots \quad (22),$$

or, as is in general better for calculation,

$$\tan \theta = \frac{Y}{X} \quad \ldots\ldots\ldots \quad (23),$$

whence we derive the magnitude of the resultant,

$$R = X \sec \theta \quad \ldots\ldots \quad (24).$$

The calculation will in general be facilitated by the use of logarithms; for which purpose Equations (23) and (24) are to be modified in the following manner :—

$$\text{tab. log. tan } \theta = \text{log. Y} - \text{log. X} + 10 \quad . \quad . \quad (25),$$

$$\text{log. R} = \text{log. X} + \text{tab. log. sec } \theta - 10 \quad . \quad . \quad (26).$$

Remark 1.—It is to be observed that the sums X of the different components $X_1$, $X_2$, &c., and Y of $Y_1$, $Y_2$, &c., are got by an algebraic addition, whatever may be the algebraic signs of the several terms. If the given forces act all round the point M, it will happen in the resolution that the different components do not all act in the same directions along X X′ and Y Y′. It will be necessary, therefore, to fix upon one direction as positive. Thus, if M X and M Y be positive directions, M X′, M Y′, will be negative; and absolute values of the components, which act from M to X′, and from M to Y′, must be subtracted from, instead of added to, those along M X and M Y. (§ 62).

Remark 2.—In choosing the axes of resolution, it simplifies the problem to fix on one of the lines which represent the forces, as one of the axes, and a line perpendicular to it, as the other.

Let M $L_1$, the line of the first force $P_1$ be the axis M X, and M Y, a line perpendicular to it, the other,

$\alpha_1$ in this case is nothing; and the angle $P_2 M P_1 = \alpha_2$.

Hence, if $\alpha_1 = 0$, the resolution of the first force is

$$P_1 \begin{cases} X_1 = P_1 \cos \alpha_1 = P_1 \\ Y_1 = P_1 \sin \alpha_1 = 0, \end{cases}$$

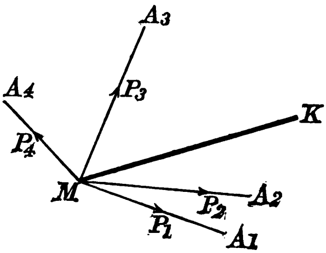
that is, $P_1$ requires no resolution.

If two of the forces happen to be at right angles, it will be convenient to choose the axes along them, and then neither requires resolution.

Actual cases may often be simplified by observing if any two of the forces are opposite, in which case, one force, equal to the excess of the greater above the less, and acting in the direction of the greater, may be taken instead of them.

Remark 3.—When the direction of the resultant is known, and its magnitude is required, it is most convenient to make it one of the axes of resolution.

Fig. 30.



Let M K (fig. 30) be the direction of the resultant of $P_1$, $P_2$, $P_3$, $P_4$, the different forces. Resolve each force into two, one along M K, and one in a line perpendicular to it. Add the components along M K. The sum must be the magnitude of the resultant; and the components along the other line must balance one another. Hence,

$$X = R = P_1 \cos A_1 M K + P_2 \cos A_2 M K + \&c.,$$

and $$Y = P_1 \sin A_1 M K + P_2 \sin A_2 M K + \&c. = 0.$$

Remark 4.—Equations (23) and (24) may be employed with advantage in all cases where the numbers of significant figures in the values to be used for X and Y are large.

By equations (23) and (24) the direction of the resultant is first determined, and then its magnitude, not as in equations (21) and (22), the magnitude first, and then the direction.

# CHAPTER V.

## GEOMETRICAL DIGRESSION.—PROJECTIONS.—SPECIFICATION OF POINTS AND LINES.

§ 97. The projection of a point on a straight line, is the point in which the latter is cut by a perpendicular to it from the former.

§ 98. Any line, joining two points, is called an arc. It is not necessary to confine this expression to its most usual signification of a continuous curve line. It may be applied to a straight line joining two points, as an extreme case; or it may be applied to a zig-zag or angular path from one point to the other; or to a self-cutting path, whether curved or polygonal; in short, to any track whatever, from one point to the other.

§ 99. The projection of an arc on a straight line, is the portion of the latter intercepted between the projections of the extremities of the former.

§ 100. If we imagine an arc divided into any number of parts; the projections of these parts, taken consecutively on any straight line, make up consecutively, the projection of the whole. Hence, the sum of the projections of the parts is equal to the projection of the whole. But in this statement, it must be understood that, of such partial projections laid down in order, those which are drawn in one direction, or *forwards*, being reckoned as positive, those which are drawn in the other direction, or backwards, must be reckoned as negative.

101. The projection of an arc on any straight line, is equal to the length of the straight line joining the extremities of the former, multiplied by the cosine of the angle* at which it is inclined to the latter. This angle, if not a right angle,

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* The angle at which one line is inclined to another, is the angle between two lines, drawn parallel to them from any point, in directions similar to the directions in the given lines which are reckoned positive.

will be acute or obtuse, according to the convention which is understood as to the direction reckoned *positive* in the line of projection; and the extremity of the arc which is taken *first* in drawing a *positive* line from one extremity of it to the other.

§ 102. The orthogonal projection of a line, straight or curved, closed or not closed, on a plane, is the locus of the points in which the latter is cut by perpendiculars to it from all points of the former.

Other kinds of projections are also used in geometry; but when no other designation is applied or understood, the simple term *projection* will always mean orthogonal projection.

§ 103. A circuit is a line returning into itself, or a line without ends in a finite space. It is (if a continuous curve) often called a "closed curve;" or if made up altogether of rectilinear parts, a "closed polygon." A circuit in one plane may be either *simple* or self-cutting. The latter variety has been called by De Morgan, "autotomic." But whether simple or autotomic, there is just one definite course to go round a circuit; and at double or multiple points, this course must be distinctly indicated,* (arrow-heads being generally used for the purpose on a diagram, like the finger-posts where two or more roads cross). A circuit not confined to one plane need never be considered to be autotomic, unless as an extreme case. Thus, if we take any thread or wire, however fine, and bend it into any curve or broken line, or tie it into the most complicated knot or succession of knots, but attach its ends together; any geometrical line drawn altogether within it, from any one point of it, round through its length back to the same point, constitutes essentially a simple or not self-cutting circuit.

§ 104. "The area enclosed by," or "the area of" a simple plane circuit, is an expression which requires no explanation. But, as has been shown by De Morgan,† a peculiar rule of interpretation is necessary to apply the same expression to an autotomic plane circuit, and it has no application, hitherto defined, to a circuit not confined to one plane.

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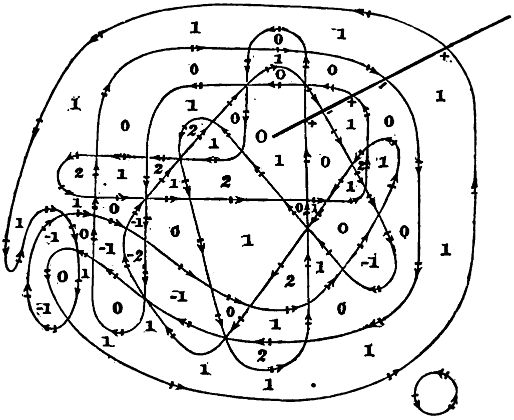
* "A curve which has double or multiple points, may be in many different ways a *circuit*, " or mode of proceeding from one point to the same again. Thus the figure of 8 may be " traced as a *self-cutting* circuit, in the way in which it is natural if the curve be a *con-* " *tinuous lemniscate*, or it may be traced as a circuit presenting two coincident salient " points. A determinate area requires a determinate mode of making the circuit." De Morgan, Cambridge and Dublin Mathematical Journal, May, 1850.

† "Extension of the word area," Cambridge and Dublin Mathematical Journal, May, 1850.

§ 105. The area of an autotomic plane circuit, is the sum of the areas of all its parts each multiplied by zero with unity as many times added as the circuit is crossed* from right to left, and unity as many times subtracted as the circuit is crossed from left to right, when a point is carried in the plane from the outside to any position within the enclosed area in question. Fig. 31, which is that given by De Morgan, will show more clearly what is meant by this use of the word area. The reader, with this as a model, may exercise himself by drawing autotomic circuits and numbering the different portions of the enclosed area according to the rule, which he will then find no difficulty in understanding.†

Fig. 31.



§ 106. Any portion of surface, edged or bounded by a circuit, is called a shell.

A plane area may be regarded as an extreme case, but generally the surface of a shell will be supposed to be curved.

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*A point is said to cross a plane circuit from right to left, if it crosses from the right side to the left side as regarded by a person looking from any point of the circuit in the direction reckoned positive.

† For an application of the principles here stated to physical investigation see the paper by Sir William Thomson in the Appendix.

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A simple shell is a shell of which the surface is single throughout. One side of the shell must always be distinguished from the other, whatever may be the convolutions of its surface. Thus we shall have a marked and unmarked side, or an outside and an inside, to distinguish from one another.

§ 107. The projection of a shell on any plane, is the area included in the projection of its bounding line.

§ 108. If we imagine a shell divided into any number of parts, the projections of these parts on any plane make up the projection of the whole. But in this statement it must be understood that the areas of partial projections are to be reckoned as positive only if the marked side, or, as we shall call it, the outside, of the projected area, and a marked side, which we shall call the front, of the plane of projection, face the same way.

If the outside of any portion of the projected area faces on the whole backwards, relatively to the front of the plane of projection, the projection of this portion is to be reckoned as negative in the sum.

Of course if the projected surface, or any part of it, be a plane area at right angles to the plane of projection, the projection vanishes.

Cor. The projections of any two shells having a common edge, on any plane, are equal. The projection of a closed surface (or a shell with evanescent edge), on any plane, is nothing.

§ 109. Equal areas in one plane or in parallel planes, have equal projections on any plane, whatever may be their figures. [The proof is easily found.]

Hence the projection of any plane figure, or of any shell, edged by a plane figure, on another plane, is equal to its area, multiplied by the cosine of the angle at which its plane is inclined to the plane of projection. This angle is acute or obtuse, according as the marked sides of the projected area, and of the plane of projection face, on the whole, towards the same parts, or on the whole oppositely.

§ 110. Two rectangles, with a common edge, but not in one plane, have their projection on any other plane, equal to that of one rectangle having their two remote sides for one pair of its opposite sides. For, the sides of this last-mentioned rectangle constitute the edge of a "shell," which we may make by applying two equal and parallel triangular areas to the sides of the given rectangles; and the sum of the projections of these two triangles on any plane, according to the rule of § 108, is nothing.

Hence (as is shown by a very simple geometrical proof, which is left as an exercise to the student), we have the following construction to find a single plane area whose projection on any plane is equal to the sum of the projections of any two given plane areas.

From any convenient point of reference draw straight lines perpendicular to the two given plane areas *forward*, relatively to their marked sides considered as fronts. Make these lines numerically equal to the two areas respectively. On these describe a parallelogram, and draw the diagonal of this parallelogram through the point of reference. Place an area with one side marked as front, in any position perpendicular to this diagonal, facing forwards, and relatively to the direction in which it is drawn from the point of reference. Make this area equal numerically to the diagonal. Its projection on any plane will be equal to the sum of the projections of the two given areas, on the same plane.

The same construction may be continued; just as, in § 74, the geometrical construction to find the resultant of any number of forces; and thus we find a single plane area whose projection on any plane is equal to the sum of the projections on the same plane of any given plane areas. And as any shell may, (if it be not composed of a finite), be regarded as composed of an infinite number of plane areas, the same construction is applicable to a shell. Hence the projection of a shell on any plane is equal to the projection on the same plane, of a certain plane area, determined by the preceding construction.

From this it appears that the projection of a shell is nothing on any plane perpendicular to the one plane on which its projection is greater than on any other; and that the projection on any intermediate plane is equal to the greatest projection multiplied by the cosine of the inclination of the plane of the supposed projection to the plane of greatest projection.

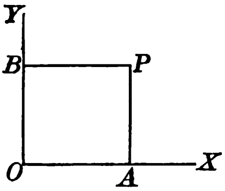
§ 111. To specify a point is to state precisely its position. As we have no conception of position, except in so far as it is relative, the specification of a point requires definite objects of reference, that is, objects to which it may be referred. The means employed for this purpose are certain elements called co-ordinates, from the system of specification which Descartes first introduced into mathematics. This system seems to have originated in the following method, for describing a curve by a table of numbers, or by an equation.

§ 112. Given a plane curve, a fixed line in its plane, and a fixed point in this line, choose as many points in the curve as are required to indicate sufficiently its form: draw perpendiculars from them to the fixed line, and measure the distances along it, cut off by these lines, reckoning from the fixed point. In this way any number of points in the curve were specified. The parts thus cut off along the fixed line, were termed *lineae abscissae*, and the perpendiculars, *lineae ordinatim applicatae.* The system was afterwards improved by drawing through the point of reference a line at right angles to the first, and measuring off along it, the *ordinates* of the curve. The two lines at right angles to one another are called the axes of reference, or the lines of reference. The ordinate and abscissa of any point are termed its *co-ordinates:* and an equation between them, by which either may be calculated when the other is given, expresses the curve in a perfectly full and precise manner.

§ 113. It is not necessary that the lines of reference be chosen at right angles to each other. But when they are chosen, inclined at any other angle than a right angle, the co-ordinates of the point specified are not its perpendicular distances from them, but its distances from either, measured parallel to the other. Such oblique co-ordinates are sometimes convenient, but rectangular co-ordinates are, in general, the most useful; these we shall now consider.

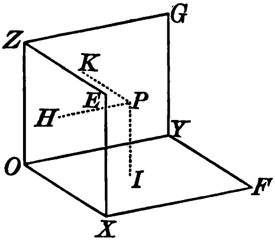
§ 114. If the points to be specified are all in one plane, the objects of reference are two lines at right angles to one another in that plane. Thus, let P (fig. 32) be a point in a plane X O Y; and let O X, O Y, be two lines in the plane, cutting each other at right angles in the point O. Then will the position of the point P be known, if the perpendicular distance of the point P from the line O X, namely, the length of the line P A, and the perpendicular distance from O Y, namely, the length of the line P B, be known.

Fig. 32.



§ 115. Again, let points, not all in one plane, but in any positions through space be considered. To specify each point now, three co-ordinates are required, and the objects of reference chosen may be three planes at right angles to one another; thus, the point P, (fig. 33) is specified by the lines P K, P H, P I, drawn perpendicular to the planes Y Z, Z X, X Y, respectively.

Fig. 33.



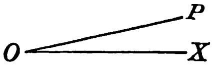
In our standard diagrams the positive directions O X, O Y, O Z, are so taken that if a watch is held in the plane X O Y, with its face towards O Z, an angular motion *against the hands*, would carry a line from O X, to O Y, through the right angle X O Y.

§ 116. When the objects to be specified are lines all passing through one point, the specifying elements employed, are angles standing in definite relation to them, and to the objects of reference. There are two chief modes in which this kind of specification is carried out: the polar and the symmetrical.

§ 117. *Polar Method.* 1°. Lines all in one plane. In this case the object of reference is any fixed line through their common point of intersection, and lying in their plane.

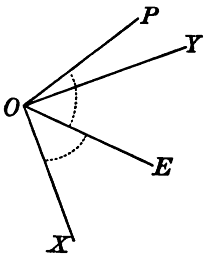
Let O (fig. 34) be the common point of intersection, O X the fixed line, and O P the line to be specified. Then the position of O P will be known, if the angle X O P, which the line O P makes with O X, be known.

Fig. 34.



2°. Lines in space, all passing through one point, may be specified by reference to a plane and a line in it, both passing through their common point of intersection.

Fig. 35.



Let O P (fig. 35) be one of a number of lines, all passing through O, to be specified with reference to the plane X O Y, and the line O X in it. Through O P let a plane be drawn, cutting the plane X O Y at right angles in O E. Then the line O P will be specified, if the angles X O E, E O P are given.

Corollary.– Similarly, if the line O P be the locus of a series of points, any one of these points will be specified, if its distance from O and the two angles specifying the line O P, are known.

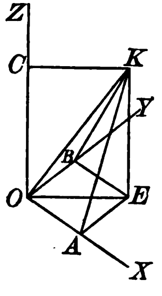
§ 118. *Symmetrical Method.* In this method the objects of reference are three lines at right angles to each other, through the common point of intersection of the lines to be specified, and the specifying elements, are the three angles which each line makes with these three lines of reference.

Thus, if O (fig. 36) be the common point of intersection, O K one of the lines to be specified, and O X, O Y, O Z, the lines of reference; then the angles X O K, Y O K, Z O K, are the specifying elements.

§ 119. From what has now been said, it will be seen that the projections of a given line on other three at right angles to each other are immediately expressible, if its direction is specified by either of the two methods.

1°. *Polar Method.* Let O K (fig. 36) be the given line, and O X, O Y, O Z, the lines along which it is to be projected. Through O Z and O K let a plane pass, cutting the plane X O Y in O E. Through K draw another plane, K E A, cutting O X perpendicularly in A and K E B cutting O Y perpendicularly in B. Then K E, being the intersection of two places each perpendicular to X O Y, is perpendicular to every line in this plane. Hence, O E K is a right angle.

Fig. 36.



Hence,

$$O E = O K \cos K O E.$$

Again, since the plane K A E was drawn perpendicular to O X, O A E is a right angle.

Hence, $\quad OA = OE \cos EOX = OK \cos KOE \cos EOX,$

and similarly, $OB = OE \cos EOY = OK \cos KOE \cos EOY,$

or if we put

$$OK = r,\ XOE = \varphi,\ EOK = i,\ KOZ = \theta = \tfrac{1}{2}\pi - i,$$

and let the required lines be denoted by $x$, $y$, $z$, then

$$\left.\begin{aligned} x &= r \sin\theta \cos\varphi. \\ y &= r \sin\theta \sin\varphi. \\ z &= r \cos\theta. \end{aligned}\right\} \quad \ldots\ldots \quad (1).$$

2°. *Symmetrical Method.* Let the line be referred to rectangular axes by the three angles, $\alpha = KOX$, $\beta = KOY$, $\gamma = KOZ$ (fig. 36).

Then the required projections are $x = r \cos\alpha$, $y = r \cos\beta$, $z = r \cos\gamma$.

§ 120. Referring again to the diagram, we have

$$OE^2 = OA^2 + OB^2,$$

and $\quad OK^2 = OE^2 + OC^2,$

therefore, $\quad OK^2 = OA^2 + OB^2 + OC^2,$

or $\quad r^2 = x^2 + y^2 + z^2 \quad \ldots\ldots\ldots \quad (2).$

Substituting here for $x, y, z$, their values, in terms of $r, \alpha, \beta, \gamma$, found above, and dividing both members of the resulting equation by $r^2$, we have

$$1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma. \quad \ldots\ldots \quad (3).$$

§ 121. In the symmetrical method, three angles are used; but, as we have seen, only two are necessary to fix the position of the line. We now see that, if two of the three angles, $\alpha, \beta, \gamma$, are given, the third can be found. Suppose $\alpha$ and $\beta$ given, then by § 120,

$$\cos^2\gamma = 1 - \cos^2\alpha - \cos^2\beta.$$

For logarithmic calculation, the following modification of the preceding formula is useful,

$$\cos^2\gamma = \sin^2\alpha - \cos^2\beta = -\cos(\alpha+\beta)\times\cos(\alpha-\beta),$$

whence $$\cos\gamma = \sqrt{\{-\cos(\alpha+\beta)\times\cos(\alpha-\beta)\}}$$

$$= \sqrt{\{\cos(\pi-\alpha-\beta)\times\cos(\alpha-\beta)\}}$$

$$\text{Tab. Log. } \cos\gamma = \tfrac{1}{2}\{\text{T.L. } \cos(\pi-\alpha-\beta)\times\text{T.L. } \cos(\alpha-\beta)\} \qquad (4).$$

§ 122. The following comparison will show in what way the two systems are related, and how it is possible to derive the specifying elements of either from those of the other. In the polar method, the fixed line in the equatorial plane, corresponds with one of the three lines of reference in the symmetrical. A line in the equatorial plane, drawn at right angles to the fixed line of the polar system, constitutes a second line of reference in the symmetrical system. The third line in the symmetrical system, is the axis of the polar system, from which the polar distance ($\theta$) is measured. A comparison of preceding formulæ shows that

$$\left.\begin{aligned} \cos\alpha &= \sin\theta\cos\varphi. \\ \cos\beta &= \sin\theta\sin\varphi. \\ \cos\gamma &= \cos\theta. \end{aligned}\right\} \quad \ldots\ldots \quad (5).$$

§ 123. The cosines of the three angles, $\alpha$, $\beta$, $\gamma$, of the symmetrical system, are commonly called the "direction cosines" of the line specified. If we denote them by $l$, $m$, $n$, we have as above,

$$l^2+m^2+n^2 = 1. \quad \ldots\ldots\ldots \quad (6).$$

A line thus specified is for brevity, called "the line $(l,m,n)$."

If $l$, $m$, $n$, are the direction cosines of a certain line; it is clear that $-l$, $-m$, $-n$, are the direction cosines of the line in the opposite direction from O. Thus it appears that the direction cosines of the line, specify not only the straight line in which it lies, but the direction in it which is reckoned as positive.

§ 124. We conclude this digression with some applications of the principles explained in it, which are useful in many dynamical investigations.

*(a.)* To find the mutual inclination, $\theta$, of two lines, $(l, m, n)$, $(l', m', n')$. Measure off any length O K $= r$, along the first line, (see fig. 36). We have, as above

$$\mathrm{O\,A} = l\,r,\ \mathrm{A\,E} = m\,r,\ \mathrm{E\,K} = n\,r.$$

Now (§ 100), the projection of O K on the second line, is equal to the sum of the projections of O A, A E, E K, on the same. But the cosines of the angles at which these several lengths are inclined to the line of projection, are respectively $\cos\theta$, $l'$, $m'$, $n'$. Hence

$$\mathrm{O\,K}\cos\theta = \mathrm{O\,A}\,l' + \mathrm{A\,E}\,m' + \mathrm{E\,K}\,n'.$$

If we substitute in this, for O K, O A, A E, E K, their values shown above, and divide both members by $r$, it becomes

$$\cos\theta = l\,l' + m\,m' + n\,n' \quad . \quad . \quad . \quad . \quad . \quad (7),$$

a most important and useful formula.

Sometimes it is useful to have the sine instead of the cosine of $\theta$. To find it we have of course,

$$\sin^2\theta = 1 - (l\,l' + m\,m' + n\,n')^2.$$

This expression may be modified thus:—instead of 1, take what is equal to it,

$$(l'^2 + m'^2 + n'^2)\,(l^2 + m^2 + n^2),$$

and the second member of the preceding becomes

$$(l^2 + m^2 + n^2)\,(l'^2 + m'^2 + n'^2) - (l\,l' + m\,m' + n\,n')^2$$

$$= (m\,n')^2 + (n\,m')^2 - 2\,m\,m'\,n\,n' + \&\text{c.}, = (m\,n' - n\,m')^2 + \&\text{c.}$$

Hence,

$$\sin\theta = \left\{ (m\,n' - n\,m')^2 + (n\,l' - l\,n')^2 + (l\,m' - m\,l')^2 \right\}^{\frac{1}{2}} \,..(8).$$

*(b.)* To find the direction cosines, $\lambda$, $\mu$, $\nu$, of the common perpendicular to two lines, $(l, m, n)$, $(l', m', n')$.

The cosine of the inclination of $(\lambda, \mu, \nu)$ to $(l, m, n)$ is, according to (7) above, $l\lambda + m\mu + n\nu$; and therefore

$$\left.\begin{array}{lr} & l\lambda + m\mu + n\nu = 0, \\ \text{similarly,} & l'\lambda + m'\mu + n'\nu = 0, \\ \text{also (§ 123)} & \lambda^2 + \mu^2 + \nu^2 = 1. \end{array}\right\} \quad . \; . \; . \quad (9).$$

These three equations suffice to determine the three unknown quantities, $\lambda$, $\mu$, $\nu$. Thus, from the first two of them, we have

$$\frac{\lambda}{mn' - nm'} = \frac{\mu}{nl' - ln'} = \frac{\nu}{lm' - l'm} \quad . \; . \quad (10).$$

From these and the third of (9), we conclude

$$\lambda = \left\{ \frac{mn' - nm'}{(mn' - nm')^2 + (nl' - ln')^2 + (lm' - l'm)^2} \right\}, \text{ \&c.,}$$

or if we denote, as above, by $\theta$, the mutual inclination of $(l, m, n)$, $(l', m', n')$;

$$\lambda = \frac{(mn' - nm')}{\sin\theta}, \; \mu = \frac{(nl' - ln')}{\sin\theta}, \; \nu = \frac{(lm' - ml')}{\sin\theta}. \quad . \; . \quad (11).$$

The sign of each of these three expressions may be changed, in as much as either sign may be given to the numerical value found for $\sin\theta$ by (8). But as they stand, if $\sin\theta$ is taken positive, they express the direction cosines of the perpendicular drawn from O through the face of a watch, held in the plane $(l\ m\ n)$, $(l'\ m'\ n')$, and so facing that an angular motion, *against or with the hands*, would carry a line from the direction, $(l, m, n)$, through an angle less than 180° to the direction, $(l,'\ m,'\ n')$, according as angular motion, through a right angle from O X to O Y is *against or with the hands* of a watch, held in the plane XOY, and facing towards OZ. This rule is proved by supposing, as a particular case, the lines $(l, m, n)$, $(l', m', n')$, to coincide with O X and O Y respectively; and then supposing them altered in their mutual inclination to any other angle between 0 and $\pi$, and their plane turned to any position whatever.

If we measure off any lengths, $\mathrm{O\,K} = r$, and $\mathrm{O\,K}' = r'$, along the two lines, $(l, m, n)$, and $(l', m', n')$, and describe a parallelogram upon them, its area is equal to $rr' \sin\theta$, since $r' \sin\theta$ is the length of the perpendicular from K′ to O K. Hence, using the preceding expression (8) for sin $\theta$, and taking

$$lr = x,\ mr = y,\ nr = z,$$
$$l'r' = x',\ m'r' = y',\ n'r' = z',$$

we conclude the following propositions.

*(c.)* The area of a parallelogram described upon lines from the origin of co-ordinates to points $(x, y, z)$, $(x', y', z')$ is equal to

$$\left\{ (yz' - zy')^2 + (zx' - z'x)^2 + (xy' - x'y)^2 \right\}^{\frac{1}{2}} . \quad (12).$$

And, as $\lambda$, $\mu$, $\nu$, are the cosines of the angles at which the plane of this area is inclined to the planes of Y Z, Z X, X Y, respectively, its projections on these planes are

$$yz' - zy',\ zx' - z'x,\ xy' - x'y. \quad . \quad . \quad . \quad (13).$$

The figures of these projections are parallelograms in the three planes of reference; that in the plane Y Z, for instance, being described on lines drawn from the origin to the points $(y, z)$, and $(y', z')$. It is easy to prove this (and, of course, the corresponding expressions for the two other planes of reference), by elementary geometry. Thus, it is easy to obtain a simple geometrical demonstration of the equations (8) and (11). It is sufficient here to suggest this investigation as an exercise to the student. It essentially and obviously includes the *rule of signs*, stated above (§ 124 (*a*)).

*(d.)* The volume of a parallelepiped described on O K, O K′, O K″, three lines drawn from O to three points $(x, y, z)$, $(x', y', z')$, $(x'', y'', z'')$, is equal to

$$x''(yz' - y'z) + y''(zx' - z'x) + z''(xy' - x'y) \ . \ (14),$$

an expression which is essentially positive, if O K, O K′, O K″, are arranged in order similarly to O X, O Y, O Z, (see § 115 above). The proof is left as an exercise for the student.

In modern algebra, this expression is called a *determinant*, and is written thus:—

$$\begin{vmatrix} x, & y, & z, \\ x', & y', & z', \\ x'', & y'', & z'', \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15).$$

# CHAPTER VI.

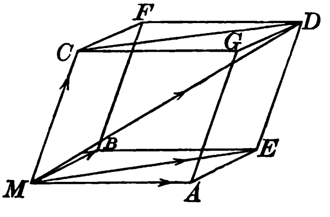
## EQUILIBRIUM OF FORCES NOT IN ONE PLANE.—PARALLELEPIPED OF FORCES.—APPLICATION OF TRIGONOMETRY.—TRIPLE RECTANGULAR RESOLUTION.—GENERAL PROBLEM OF COMPOSITION.—EQUATIONS OF EQUILIBRIUM.

§ 125. We have already seen (§ 73) how the resultant of any number of forces, whether they are in one plane or not, may be obtained by a geometrical construction founded on successive applications of the parallelogram of forces. The case of three forces in different planes may be particularly considered. It is usually called "the Parallelepiped of Forces," on account of the solid figure which occurs in connection with its geometrical solution. We give it in particular here, rather, however, because it is introductory to triple rectangular resolution, than because it involves any new principle.

§ 126. Parallelepiped of Forces.—The resultant of three forces acting on a material point, in lines which are not in the same plane, is represented in magnitude and direction, by the diagonal drawn from the point to the remote solid angle of the parallelepiped described on the lines which represent them.

Let M (fig. 37) be a material point, acted on by forces represented by the lines M A, M B, M C. Through A, draw a plane parallel to B M C; through B, draw a plane parallel to C M A; and through C, draw a plane parallel to A M B. Let these three planes meet in D. Join M D. This line represents the resultant in position and magnitude.

Fig. 37.

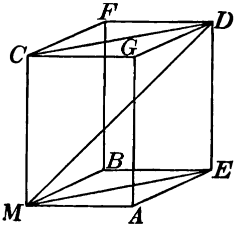


This proposition is an obvious consequence of the parallelogram of forces, as we may see by considering the plane figures M A E B, and M E D C, which are easily proved to be parallelograms.

§ 127. The simple trigonometrical formulæ already given (§ 80, Eq. 1, 2, 3), may be applied directly to forces not in one plane; first to find the resultant of any two of them, then the resultant of the resultant thus obtained and another force, and so on until there are no more forces to compound. This process, which would become very cumbrous if the forces were numerous, and their magnitudes and relative positions specified by means of numerical data, may be simplified by reducing the given forces to three along lines at right angles to each other, and finding the resultant of these three. We have evidently here, as in the previous case of forces in one plane (§ 79), two classes of problems to deal with;—those concerning the resolution of forces along three rectangular axes, and those concerning the composition of forces given at right angles to one another. We commence with the last mentioned.

§ 128. To find the resultant of three forces acting on a material point in lines at right angles to one another.

Fig. 38.



1°. To find the magnitude of the resultant. Let the forces be given numerically, X, Y, Z, and let them be represented respectively by the lines MA, MB, MC (fig. 38) at right angles to one another.

First determine the resultant of X and Y in magnitude. If we denote it by R′, we have (§ 90)

$$R' = \sqrt{(X^2 + Y^2)}. \quad . \quad . \quad . \quad . \quad . \quad (1).$$

This resultant, represented by M E (§ 66), lies in the plane B M A; and since the lines of the forces X and Y are perpendicular to the line M C, the line M E must also be perpendicular to it; for, "if a line be perpendicular to two other lines, it is perpendicular to every other line in their plane;" hence R′ acts perpendicularly to Z.

Next, find the resultant of R′ and Z, the third force. If we denote it by R, we have

$$R = \sqrt{(R'^2 + Z^2)},$$

and substituting for R′² its value, we have

$$R = \sqrt{(X^2 + Y^2 + Z^2)}. \quad . \quad . \quad . \quad . \quad (2).$$

2°. To find the direction of the resultant. Determine first the inclination of the subsidiary resultant R′ to M A or M B. Let the angle E M A be denoted by $\varphi$; then we have

$$\cos \varphi = \frac{X}{R'}.$$

Next, let $\gamma$ denote the angle at which the line M D is inclined to M C; that is, the angle C M D; we have

$$\cos \gamma = \frac{Z}{R} \quad \ldots\ldots\ldots \quad (3).$$

Thus, by means of the two angles $\gamma$ and $\varphi$, the position of the line MD, and, consequently, that of the resultant is found.

§ 129. In the numerical solution of actual cases, it will generally be found most convenient to calculate the three elements in the following order: 1°, $\varphi$, 2°, $\gamma$, 3°, R.

1°. To calculate $\varphi$, the formula already given, may be taken

$$\tan \varphi = \frac{Y}{X}. \quad \ldots\ldots\ldots \quad (4).$$

2°. To calculate $\gamma$. We have

$$\tan \gamma = \frac{R'}{Z}. \quad \ldots\ldots\ldots \quad (5).$$

But

$$R' = X \sec \varphi$$

Hence

$$\tan \gamma = \frac{X \sec \varphi}{Z}. \quad \ldots\ldots \quad (6).$$

3°. To calculate R.

$$R = Z \sec \gamma. \quad \ldots\ldots\ldots \quad (7).$$

§ 130. The angles determined by these equations specify the line of the resultant, by what was called in a previous Chapter (§ § 117, 119) the Polar method.

The symmetrical specification of the resultant is to be found thus:—Let (fig. 38) the angles at which the line of the resultant, M D, is inclined to those of the forces be respectively denoted by $\alpha$, $\beta$, and $\gamma$. Then, as above (Eq. 3)

$$\cos \gamma = \frac{Z}{R} \quad \ldots\ldots\ldots \quad (8).$$

By the same method we shall find

$$\cos\alpha = \frac{X}{R} \qquad (9),$$

and

$$\cos\beta = \frac{Y}{R} \qquad (10).$$

If, therefore, there are three forces at right angles to one another, the cosine of the inclination of their resultant to any one of them is equal to this force divided by the resultant.

This method requires that the magnitude of the resultant be known before its position is determined. For the latter purpose, any two of the angles, as was shown in Chapter V, are sufficient.

§ 131. We shall now consider the resolution of forces along three specified lines. The most important case of all is that in which the lines are at right angles to one another.

Let the force R, given to be resolved, be represented by M D (fig. 38), and let the angles which it forms with the lines of resolution be given, either $\alpha$, $\beta$, $\gamma$, or $\gamma$, $\varphi$. Required the components, X, Y, Z.

1°. Suppose $\alpha$, $\beta$, $\gamma$ are given, then we deduce from Eq. (9)

$$X = R\cos\alpha;$$

from Eq. (10)

$$Y = R\cos\beta;$$

and from Eq. (8)

$$Z = R\cos\gamma.$$

2°. Suppose the data are R, $\gamma$, $\varphi$, that is, the magnitude of the resultant, its inclination to one of the axes of resolution, and the inclination of the plane of the resultant and that axis to either of the other axes.

To find the components X and Y: resolve the force R in the vertical plane C M E D (fig. 38) into two rectangular components along M C and M E. Let the angle C M D be denoted by $\gamma$. Then we have for the component along M C

$$Z = R\cos\gamma, \qquad (11),$$

and for the component along M E

$$ME = R\sin\gamma.$$

Next, resolve the component along M E in the horizontal plane B M A E, into two, one along M A, and the other along M B. Let the angle E M A be denoted by $\varphi$. Then we have for the component along M A,

$$X = M\,E \cos \varphi = R \sin \gamma \cos \varphi, \quad . \quad . \quad . \quad (12),$$

and for the component along M B,

$$X = M\,E \sin \varphi = R \sin \gamma \sin \varphi. \quad . \quad . \quad . \quad (13).$$

§ 132. We are now prepared to solve the general problem:—Given, any number of forces acting on one point, in lines which lie in different planes, required their resultant in position and magnitude.

Through the point acted on, draw three lines or axes of resolution at right angles to one another. Resolve each force, by § 131, 1°, or by § 131, 2°, into three components, acting respectively along the three lines. When all the forces have been thus treated, add severally the sets of components: by this means, all the forces are reduced to three at right angles to one another. Find, by Eq. 2, their resultant: the single force thus obtained is the resultant of the given forces, which was to be found.

Remark.—All the remarks made with reference to the resolution and composition of forces along two axes (§ 96) apply, with the necessary extension, to that of forces along three.

§ 133. We are now prepared to answer the question which forms the first general head of Statics (§ 27); "What are the conditions of Equilibrium of a material point?" The answer may be put in one or other of two forms.

1°. If a set of forces acting on a material point be in equilibrium, any one of them must be equal and opposite to the resultant of the others: or,

2°. If a set of forces acting on a material point be in equilibrium, the resultant of the whole set must be equal to nothing.

§ 134. Let us consider the first of these statements.

Given, a set of forces, $P_1$, $P_2$, $P_3$, &c., in equilibrium: the force $P_1$, for example, is equal and opposite to the resultant of $P_2$, $P_3$, &c.; or, the resultant of $P_2$, $P_3$, &c., is $-P_1$. Omitting

$P_1$, find the resultant of the remaining forces by the general method; the components of this resultant will be

$$P_2 \cos \alpha_2 + P_3 \cos \alpha_3 + \&c. \text{ along } MX.$$

$$P_2 \cos \beta_2 + P_3 \cos \beta_3 + \&c. \text{ along } MY.$$

$$P_2 \cos \gamma_2 + P_3 \cos \gamma_3 + \&c. \text{ along } MZ.$$

Now, if $-P_1$ be the resultant, the components of $-P_1$ will be equivalent respectively to the components of this resultant, therefore

$$-P_1 \cos \alpha_1 = P_2 \cos \alpha_2 + P_3 \cos \alpha_3 + \&c.$$

$$-P_1 \cos \beta_1 = P_2 \cos \beta_2 + P_3 \cos \beta_3 + \&c.$$

$$-P_1 \cos \gamma_1 = P_2 \cos \gamma_2 + P_3 \cos \gamma_3 + \&c.$$

Which equations, in the following more general form, express the required conditions:

$$P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + P_3 \cos \alpha_3 + \&c. = 0.$$

$$P_1 \cos \beta_1 + P_2 \cos \beta_2 + P_3 \cos \beta_3 + \&c. = 0.$$

$$P_1 \cos \gamma_1 + P_2 \cos \gamma_2 + P_3 \cos \gamma_3 + \&c. = 0.$$

§ 135. The second form of the answer may be illustrated either *a*, dynamically, or *b*, algebraically.

*a.* Suppose all the forces reduced to three, X, Y, Z, acting at right angles to each other. Under what circumstances will three forces give a vanishing resultant? Substitute for X and Y their resultant $R'$, and consider $R'$ and Z at right angles to one another. If they give a vanishing resultant, that is, if Z and $R'$ balance, they must either be equal and directly opposed (Ax. III), or else they must each be equal to nothing. But they are not directly opposed, therefore each is equal to nothing. Now, since $R' = 0$, X and Y, which are equivalent to $R'$, must also each be equal to nothing: in order, therefore, that the resultant of forces acting along three lines at right angles to one another may vanish, we have

$$P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \&c. = 0.$$

$$P_1 \cos \beta_1 + P_2 \cos \beta_2 + \&c. = 0.$$

$$P_1 \cos \gamma_1 + P_2 \cos \gamma_2 + \&c. = 0.$$

*b.* The general expression for the resultant is

$$R^2 = X^2 + Y^2 + Z^2.$$

Now, for equilibrium,

$$R = 0.$$

and therefore,

$$X^2 + Y^2 + Z^2 = 0.$$

But the sum of three positive quantities can be equal to nothing, only when each of them is nothing: hence

$$X = 0,$$

$$Y = 0,$$

$$Z = 0.$$

## DIVISION II.

§ 136. Under the second general head of statics, we have to consider forces acting on a rigid body, or a system of material points rigidly connected. (§ § 27, 29). The subject may be divided as follows:—

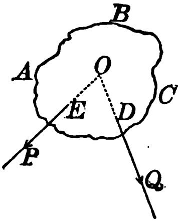
Forces acting in a plane in lines which meet.
Two parallel forces.
Any number of parallel forces in one plane.
Any number of parallel forces not in one plane.
Couples.
Forces in general, acting on a rigid body, whether in one plane or not.

# CHAPTER VII.

## FORCES IN ONE PLANE, IN LINES WHICH MEET.—FORCES IN PARALLEL LINES.

§ 137. Forces whose lines meet. Let A B C (fig. 39) be a rigid body acted on by two forces, P and Q, applied to it at different points, D and E respectively, in lines in the same plane.

Fig. 39.



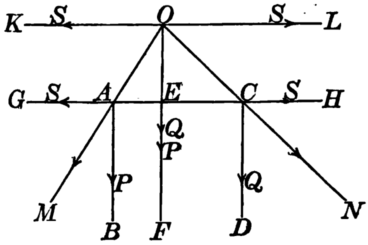
Since the lines are not parallel, they will meet if produced; let them be produced and meet in O. Transmit the forces to act on that point; and the result is that we have simply the case of two forces acting on a material point, which has been already considered.

§ 138. The preceding solution is applicable to every case of non-parallel forces in a plane, however far removed the point may be in which their lines of action meet, and the resultant will of course be found by the parallelogram of forces. The limiting case of parallel forces, or forces whose lines of action, however far produced, do not meet, was considered in § 88 above, and the position and magnitude of the resultant were investigated. The following is an independent demonstration of the conclusion there arrived at.

§ 139. Parallel forces in a plane. The resultant of two parallel forces is equal to their sum, and is in the parallel line which divides any line drawn across their lines of action into parts inversely as their magnitudes.

1°. Let P and Q (fig. 40) be two parallel forces acting on a rigid body in similar directions in lines A B and C D. Draw any line A C across their lines. In it introduce any pair of balancing forces, S in A G and S in C H. These forces will not disturb (Ax. I) the equilibrium of the body. Suppose the forces P and S in A G, and Q and S in C H to act respectively on the points A and C of the rigid body. The forces P and S, in A B and A G, have a single resultant in some line A M, within the angle G A B; and Q and S in C D and C H have a resultant in some line C N, within the angle D C H.

Fig. 40.



The angles M A C, N C A are together greater than two right angles, hence the lines M A, N C will meet if produced. Let them meet in O. Now the two forces P and S may be transferred (Preliminary Proposition III) to parallel lines through O. Similarly the forces Q and S may be also transferred. Then there are four forces acting on O, two of which, S in O K and S in O L, are equal and directly opposed. They may, therefore, be removed, and there are left two forces equal to P and Q in one line on O, which are equivalent to a single force P + Q in the same line.

2°. If, for a moment, we suppose O E to represent the force P, then the force representing S must be equal and parallel to E A, since the resultant of the two is in the direction O A. That is to say,

$$S : P :: EA : OE;$$

And in like manner, by considering the forces S in O L and Q in O E, we find that

$$Q : S :: OE : EC.$$

Compounding these analogies, we get

$$\frac{S}{P} \times \frac{Q}{S} = \frac{EA}{OE} \times \frac{OE}{EC}$$

whence

$$\frac{Q}{P} = \frac{EA}{EC},$$

or

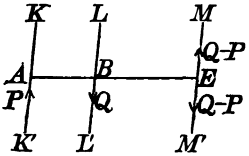
$$Q : P :: EA : EC,$$

that is, the parts into which the line is divided by the resultant are inversely as the forces.

§ 140. Forces in dissimilar directions. The resultant of two parallel forces in dissimilar directions,* of which one is greater than the other, is found by the following rule: Draw any line across the lines of the forces and produce it across the line of the greater, until the whole line is to the part produced as the greater force is to the less; a force equal to the excess of the greater force above the less, applied at the extremity of this line in a parallel line and in the direction similar to that of the greater, is the resultant of the system.

Let P and Q in K K′ and L L′ (fig. 41), be the contrary forces. From any point A, in the line of P, draw a line A B across the line of Q cutting it in B, and produce the line to E, so that A E : B E :: Q : P. Through E draw a line M M′ parallel to K K′ or L L′.

Fig. 41.



In M M′ introduce a pair of balancing forces each equal to Q − P. Then P in A K′ and Q − P in E M have a resultant equal to their sum, or Q. This resultant is in the line L L′; for, from the analogy,

$$AE : BE :: Q : P,$$

we have

$$AE - BE : BE :: Q - P : P,$$

or

$$AB : BE :: Q - P : P.$$

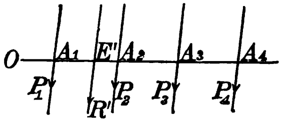
Hence P in A K′, Q in B L′, and Q−P in E M are in equilibrium and may be removed. There remains only Q − P in E M′, which is therefore the resultant of the two given forces. This fails when the forces are equal.

---

* In future the word "contrary" will be employed instead of the phrase "parallel and in dissimilar directions" to designate merely *directional opposition*, while the unqualified word "opposite" will be understood to signify *contrary and in one line.*

§ 141. Any number of parallel forces in a plane. Let $P_1$, $P_2$, $P_3$, &c., (fig. 42) be any number of parallel forces acting on a rigid body in one plane. To find their resultant in position and magnitude, draw any line across their lines of action, cutting them in points, denoted respectively by $A_1$, $A_2$, $A_3$, &c., and in it choose a point of reference O. Let the distances of the lines of the forces from this point be denoted by $a_1$, $a_2$, $a_3$, &c.; as $OA_1 = a_1$, $OA_2 = a_2$, &c. Also let R denote the resultant, and $x$ its distance from O.

Fig. 42.



Find the resultant of any two of the forces, as $P_1$ and $P_2$, by § 139. Then if we denote this resultant by R′, we have

$$R' = P_1 + P_2.$$

Divide $A_1A_2$ in E′ into parts inversely as the forces, so that

$$P_1 \times A_1E' = P_2 \times E'A_2.$$

Hence if we denote O E′ by $x'$ we have

$$P_1 \times (x' - a_1) = P_2 \times (a_2 - x')$$

or

$$(P_1 + P_2)\,x' = P_1a_1 + P_2a_2,$$

that is

$$R'\,x' = P_1a_1 + P_2a_2.$$

Similarly we shall find the resultant of R′ and $P_3$ to be

$$R'' = R' + P_3 = P_1 + P_2 + P_3;$$

and

$$R''x'' = R'x' + P_3a_3 = P_1a_1 + P_2a_2 + P_3a_3.$$

Hence, finally we have

$$R = P_1 + P_2 + P_3 + \dots\dots\dots + P_n \quad . \quad . \ (1)$$

and

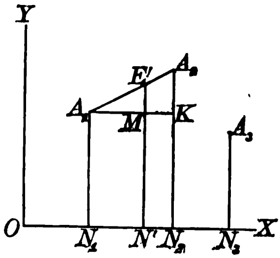
$$Rx = P_1a_1 + P_2a_2 + P_3a_3 + \dots\dots + P_na_n \quad . \ (2).$$

In this method negative forces or negative values of any of the quantities $a_1$, $a_2 \ldots$, may be included, provided the generalized rules of multiplication and addition in algebra are followed.

§ 142. Any number of parallel forces not in one plane. To find the resultant, let a plane cut the lines of all the forces, and let the points in which they are cut, be specified by reference to two rectangular axes in the plane. Let the plane be Y O X, (fig. 43) O X, O Y, the axes of reference, O the origin of co-ordinates, and $A_1$, $A_2$, $A_3$, &c., the points in which the plane cuts the lines of the forces, $P_1$, $P_2$, $P_3$, &c. Thus each of these points will be specified by perpendiculars drawn from them to the axis. Let the co-ordinates of the point $A_1$ be denoted by $x_1\ y_1$; of $A_2$, by $x_2\ y_2$; and so on; that is, $O N_1 = x_1$, $N_1 A_1 = y_1$; $O N_2 = x_2$, $N_2 A_2 = y_2$; &c., let also the final resultant be denoted by R, and its co-ordinates by $x$ and $y$.

Find the resultant of $P_1$ and $P_2$ by joining $A_1\ A_2$, and dividing the line inversely as the forces. Suppose E′ the point in which this resultant cuts the plane of reference. Then

Fig. 43.



$$P_1 \times A_1E' = P_2 \times E'A_2.$$

To find the co-ordinates, which may be denoted by $x'\ y'$, of the point E′ with reference to O X and O Y; draw E′ N′ perpendicular to O X and cutting it in N′, and from $A_1$ draw $A_1K$ parallel to O X, or perpendicular to $A_2N_2$, and cutting it in K and E′ N′ in M. Then (Euclid VI. 2)

$$A_1E' : E'A_2 :: A_1M : MK.$$

Hence

$$P_1 \times A_1M = P_2 \times MK,$$

or

$$P_1\ (x' - x_1) = P_2\ (x_2 - x'),$$

whence we get

$$(P_1 + P_2)\ x' = P_1x_1 + P_2x_2\ ;$$

and since

$$R' = P_1 + P_2,$$

we have

$$R'x' = P_1x_1 + P_2x_2,$$

and, similarly,

$$R'y' = P_1y_1 + P_2y_2.$$

We may find the resultant of $R'$ and $P_3$ in like manner, and so with all the forces. Hence we have for the final resultant,

$$R = P_1 + P_2 + P_3 + \ldots\ldots\ldots\ldots + P_n \quad . \quad . \quad . \quad (3),$$

$$Rx = P_1x_1 + P_2x_2 + P_3x_3 + \ldots + P_nx_n \quad . \quad . \quad (4),$$

$$Ry = P_1y_1 + P_2y_2 + P_3y_3 + \ldots + P_ny_n \quad . \quad . \quad (5).$$

These equations may include negative forces, or negative co-ordinates.

§ 143. Conditions of equilibrium of any number of parallel forces. In order that any given parallel forces may be in equilibrium, it is not sufficient alone, that their algebraic sum, be equal to zero.

For, let $R = P_1 + P_2 + \&c. = 0.$

From this equation it follows that if the forces be divided into two groups, one consisting of the forces reckoned positive, the other of those reckoned negative, the sum, or resultant (§ 141) of the former is equal to the resultant of the latter; that is, if ${}_{\prime}R$ and ${}'R$ denote the resultants of the positive and negative groups respectively,

$${}_{\prime}R = {}'R.$$

But unless these resultants are directly opposed they do not balance one another; wherefore, if $({}_{\prime}x{}_{\prime}y)$ and $({}'x{}'y)$ be the coordinates of ${}_{\prime}R$ and ${}'R$ respectively, we must have for equilibrium

$${}_{\prime}x = {}'x$$

and

$${}_{\prime}y = {}'y;$$

whence we get

$${}_{\prime}R_{\prime}x - {}^{\prime}R^{\prime}x = 0$$

and

$${}_{\prime}R_{\prime}y - {}^{\prime}R^{\prime}y = 0.$$

But ${}_{\prime}R_{\prime}x$ is equal to the sum of those of the terms $P_1x_1$, $P_2x_2$, &c., which are positive, and ${}^{\prime}R^{\prime}x$ is equal to the sum of the others each with its sign changed: and so for ${}_{\prime}R_{\prime}y$ and ${}^{\prime}R^{\prime}y$. Hence the preceding equations are equivalent to

$$P_1x_1 + P_2x_2 + \dots\dots\dots\dots + P_nx_n = 0.$$

$$P_1y_1 + P_2y_2 + \dots\dots\dots\dots + P_ny_n = 0.$$

We conclude that, for equilibrium, it is necessary and sufficient that each of the following three equations be satisfied :—

$$P_1 + P_2 + P_3 + \dots\dots\dots\dots + P_n = 0. \quad . \quad . \quad (6),$$

$$P_1x_1 + P_2x_2 + P_3x_3 + \dots\dots + P_nx_n = 0. \quad . \quad . \quad (7),$$

$$P_1y_1 + P_2y_2 + P_3y_3 + \dots\dots + P_ny_n = 0. \quad . \quad . \quad (8).$$

§ 144. If Eq. 6 do not hold, but Eq. 7 and 8 do, the forces have a single resultant through the origin of co-ordinates. If Eq. 6 and either of the other two do not hold, there will be a single resultant in a line through the corresponding axis of reference, the co-ordinates of the other vanishing. If Eq. 6 and either of the other two do hold, the system is reducible to a single couple in a plane through that line of reference for which the sum of the products is not equal to nothing. If the plane of reference is perpendicular to the lines of the forces the moment of this couple is equal to the sum of the products not equal to nothing.

# CHAPTER VIII.

## COUPLES.—GENERAL EQUILIBRIUM OF A RIGID BODY.

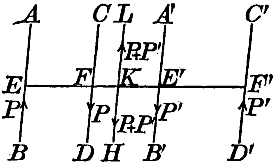
§ 145. In finding the resultant of two contrary forces in any case in which the forces are unequal—the smaller the difference of magnitude between them, the farther removed is the point of application of the resultant. When the difference is nothing, the point is removed to an infinite distance, and the construction is thus rendered nugatory, (§ 140). The general solution gives in this case $R = 0$; yet the forces are not in equilibrium, since they are not directly opposed. Hence two equal contrary forces neither balance, nor have a single resultant. It is clear that they have a tendency to turn the body to which they are applied. This system was by Poinsot denominated a couple.

In actual cases the direction of a couple is generally reckoned positive, if the couple tends to turn contrary to the hands of a watch as seen by a person looking at its face, negative, when it tends to turn with the hands. Hence the axis, which may be taken to represent a couple, will show, if drawn according to the rule given in § 40, whether the couple is positive or negative, according to the side of its plane from which it is regarded.

§ 146. Proposition I. Any two couples in the same or in parallel planes are in equilibrium if their moments are equal and they tend to turn in contrary directions.

1°. Let the forces of the first couple be parallel to those of the second, and let all four forces be in one plane.

Fig. 44.



Let (fig. 44) the forces of the first couple be P in A B and C D, and of the second P′ in A′ B′ and C′ D′. Draw any line E F′ across the lines of the forces, cutting them respectively in points E, F, E′ and F′; then the moment of the first couple is $P.EF$ and of the second $P'.E'F'$; and since the moments are equal we have

$$P.EF = P'.E'F'.$$

Of the four forces, P in A B and P′ in C′D′ act in similar directions, and P in C D and P′ in A′B′ also act in similar directions; and their resultants respectively can be determined by the general method, § 141. The resultant of P in A B, and P′ in C′D′, is thus found to be equal to $P + P'$, and if H L is the line in which it acts,

$$P.EK = P'.KF',$$

Again, we have

$$P.EF = P'.E'F'.$$

Subtract the first member of the latter equation from the first member of the former, and the second member of the latter from the second member of the former: there remains

$$P.FK = P'.KE',$$

from which we conclude, that the resultant of P in C D and P′ in A′B′ is in the line L H. Its magnitude is $P + P'$. Thus the given system is reduced to two equal resultants acting in opposite directions in the same straight line. These balance one another, and therefore the given system is in equilibrium.

Corollary. A couple may be transferred from its own arm to any other arm in the same line, if its moment be not altered.

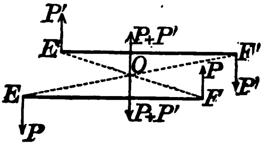
§ 147. Proposition I. 2°. All four forces in one plane, but those of one couple not parallel to those of the other.

Produce their lines to meet in four points; and consider the parallelogram thus formed. The products of the sides, each into its perpendicular distance from the side parallel to it, are equal, being the area of the parallelogram. Hence, since the moments of the two couples are equal, their forces are proportional to the sides of the parallelogram along which they act. And, since the couples tend to turn in opposite directions, the four forces represented by the sides of a parallelogram act in similar directions relatively to the angles, and dissimilar directions in the parallels, and therefore (§ § 58, 69) balance one another.

Corollary. The statical effect of a couple is not altered, if its arm be turned round any point in the plane of the couple.

§ 148. Proposition I. 3°. The two couples not in the same plane, but the forces equal and parallel.

Fig. 45.



Let there be two couples, acting respectively on arms $EF$ and $E'F'$ (fig. 45), which are parallel but not in the same plane. Join $EF'$ and $E'F$. These lines bisect one another in O.

Of the four forces, $P$ on $F$ and $P'$ on $E'$, act in similar directions, and their resultant equal to $P + P'$, may be substituted for them. It acts in a parallel line through O. Similarly $P$ on $E$ and $P'$ on $F'$ have also a resultant equal to $P + P'$ through O; but these resultants being equal and opposite, balance, and therefore the given system is in equilibrium.

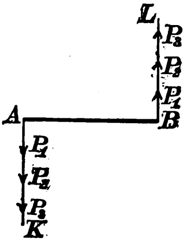
Remark 1.—A corresponding demonstration may be applied to every case of two couples, the moments of which are equal, though the forces and arms may be unequal. When the forces and arms are unequal, the lines $EF'$, $E'F$ cut one another in O into parts inversely as the forces.

Remark 2.—Hence as an extreme case, Prop. I, 1°, may be brought under this head. Let $EF$ (fig. 44) be the arm of one couple, $E'F'$ of the other, both in one straight line. Join $FE'$, and divide it inversely as the forces. Then $FK : KE' :: EF : E'F'$ and $EF'$ is divided in the same ratio.

Corollary. Transposition of couples. Any two couples in the same or in parallel planes, are equivalent, provided their moments are equal, and they tend to turn in similar directions.

§ 149. Proposition II. Any number of couples in the same or in parallel planes, may be reduced to a single resultant couple, whose moment is equal to the algebraic sum of their moments, and whose plane is parallel to their planes.

Fig. 46.

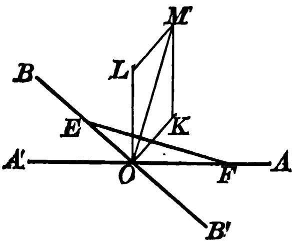


Reduce all the couples to forces acting on one arm $AB$, which may be denoted by $a$, (fig. 46). Then if $P_1$, $P_2$, $P_3$, &c., be the forces, the moments of the couples will be $P_1a$, $P_2a$, $P_3a$, &c. Thus we have

$P_1, P_2, P_3$, &c., in A K, reducible to a single force, their sum, and similarly, a single force $P_1 + P_2 +$ &c., in B L. These two forces constitute a couple whose moment is $(P_1 + P_2 + P_3 + \&c.)\, a$. But this product is equal to $P_1a + P_2a + P_3a +$ &c., the sum of the moments of the given couples, and therefore any number of couples, &c. If any of the couples act in the direction opposite to that reckoned positive, their moments must be reckoned as negative in the sum.

§ 150. Proposition III. Any two couples not in parallel planes may be reduced to a single resultant couple, whose axis is the diagonal through the point of reference of the parallelogram described upon their axes.

Fig. 47.



1°. Let the planes of the two couples cut the plane of the diagram perpendicularly in the lines A A′ and BB′ respectively (fig. 47); let the planes of the couples also cut each other in a line cutting the plane of the diagram in O. Through O, as a point of reference, draw O K the axis of the first couple, and O L the axis of the second. On O K and O L construct the parallelogram O K M L. Its diagonal O M is the axis of the resultant couple.

Let the moment of the couple acting in the plane BB′, be denoted by G, and of that in A A′, by H. For the given couples, substitute two others, with arms equal respectively to G and H, and therefore with forces equal to unity.

From O B and O A measure off O E = G, and O F = H, and let these lines be taken as the arms of the two couples respectively. The forces of the couples will thus be perpendicular to the plane of the diagram: those of the first, acting outwards at E, and inwards at O; and those of the second, outwards at O, and inwards at F. Thus, of the four equal forces which we have in all, there are two equal and opposite at O, which therefore balance one another, and may be removed; and there remain two equal parallel forces, one acting outwards at E, and the other at F inwards, which constitute a couple on an arm E F.

This single couple is therefore equivalent to the two given couples.

2°. It remains to be proved that its axis is O M. Join E F. As, by construction, O L and O K are respectively perpendicular to O A, and O B, the angle K O L is equal to the angle A O B′. Hence, M L O, the supplement of the former is equal to E O F, the supplement of the latter. But O K is equal to O E; each being equal to the moment of the first of the given couples; and therefore L M, which is equal to the former, is equal to O E. Similarly O L is equal to O F. Thus there are two triangles, M L O and E O F, with two sides of one respectively equal to two sides of the other, and the contained angles equal: therefore the remaining sides O M, E F are equal, and the angles L O M, O F E are equal. But since O L is perpendicular to O F, O M is perpendicular to E F. Hence O M is the axis of the resultant couple.

§ 151. Proposition IV. Any number of couples whatever are either in equilibrium with one another, or may be reduced to a single couple, under precisely the same conditions as those already investigated for forces acting on one point, the axes of the couples being now taken everywhere instead of the lines formerly used to represent the forces.

1°. Resolve each couple into three components having their axes along three rectangular lines of reference, O X, O Y, O Z. Add all the components corresponding to each of these three lines. Then if the resultant of all the couples whose axes are along the line O X, be denoted by L,

" " " O Y, " " " M,

" " " O Z, " " " N,

and if G be the resultant of these three, we have

$$G = \sqrt{(L^2 + M^2 + N^2)}:$$

and if $\zeta$, $\eta$, $\vartheta$, be the angles which the axis of this couple G, forms with the three axes O X, O Y, O Z, respectively, we have

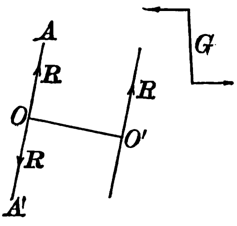
$$\cos \zeta = \frac{L}{G}; \ \cos \eta = \frac{M}{G}; \ \cos \vartheta = \frac{N}{G}.$$

§ 152. 2°. Conditions of equilibrium of any number of couples. For equilibrium the resultant couple must be equal to nothing: but as it is compounded of three subsidiary resultant couples in planes at right angles to one another, they also must each be equal to nothing. The remarks already made, and the equations already given in § § 134, 135, apply with the necessary modification to couples also. Thus, for instance, the equations of equilibrium are

$$\begin{aligned} G_1 \cos \zeta_1 + G_2 \cos \zeta_2 + G_3 \cos \zeta_3 + \&c., &= 0 \\ G_1 \cos \eta_1 + G_2 \cos \eta_2 + G_3 \cos \eta_3 + \&c., &= 0 \\ G_1 \cos \vartheta_1 + G_2 \cos \vartheta_2 + G_3 \cos \vartheta_3 + \&c., &= 0 \end{aligned}$$

§ 153. Before investigating the conditions of equilibrium of any number of forces acting on a rigid body, we shall first establish some preliminary propositions.

Fig. 48.



1°. A force and a couple in the same or in parallel planes may be reduced to a single force. Let the plane of the couple be the plane of the diagram, and let its moment be denoted by G, (fig. 48). Let R, acting in the line O A in the same plane, be the force. Transfer the couple to an arm (which may be denoted by $\alpha$) through the point O, such that each force shall be equal to R; and let its position be so chosen, that one of the forces shall act in the same straight line with R in O A, but in the opposite direction to it. R and G being known, the length of this arm can be found, for since the moment of the transposed couple is

$$R\,\alpha = G$$

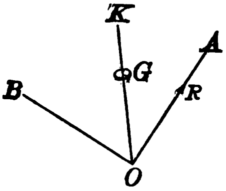
we have

$$\alpha = \frac{G}{R}.$$

Through O then, draw a line O O′ perpendicular to O A, making it equal to $\alpha$. On this arm apply the couple, a force equal to R, acting on O′ in a line perpendicular to O O′, and another in the opposite direction at the other extremity. There are now three forces, two of which, being equal and opposite to one another, in the line A A′, may be removed. One, acting on the point O′, remains, which is therefore equivalent to the given system.

§ 154. 2°. A couple and a force in a given line inclined to its plane may be reduced to a smaller couple in a plane perpendicular to the force, and a force equal and parallel to the given force.

Fig. 49.



Let O A (fig. 49) be the line of action of the force R, and let O K be the axis of the couple. Let the moment be denoted by G: and let A O K, the inclination of its axis to the line of the force be $\theta$. Draw O B perpendicular to O A. By § 151, Prop. IV, resolve the couple into two components, one acting round O A as axis, and one round O B. Thus the component round O A will be

$$G \cos \theta,$$

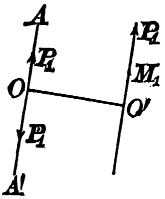
and the component round O B,

$$G \sin \theta.$$

Now as G sin $\theta$ acts in the same plane as the given force R, this component together with R may be reduced by § 153 to one force. This force which is equal to R, will act not at O in the line O A, but in a parallel line through a point O′ out of the plane of the diagram. Thus the given system is reduced to a smaller couple G cos $\theta$, and to a force in a line which, by Poinsot, was denominated the central axis of the system.

§ 155. 3°. Any number of forces may be reduced to a force and a couple.

Fig. 50.



Let $P_1$ acting on $M_1$ (fig. 50), be one of a number of forces acting in different directions on different points of a rigid body. Choose any point of reference O, for the different forces, and through it draw a line A A′ parallel to the line of the first force $P_1$. Through O, draw O O′ perpendicular to A A′ or the line of the force $P_1$. In the line A A′ introduce two equal opposite forces, each equal to $P_1$. There are now three forces, producing the same effect as the given force, and they may be grouped differently: $P_1$ acting in O in the line O A, and a couple, $P_1$ acting at O′, and $P_1$ at O in the line O A′, on an arm O O′. Reduce similarly all the other forces, each to a force acting on O, and to a couple. But all the couples thus obtained are equivalent to a single couple, and all the forces are equivalent to one force. Hence, &c.

§ 156. Reduction of any number of forces to their simplest equivalent system.

Suppose any number of forces acting in any direction on different points of a rigid body. Choose three rectangular planes of reference meeting in a point O, the origin of co-ordinates. In order to effect the reduction it is necessary to bring in all the forces to the point O. This may be done in two different ways—either in two steps, or directly.

§ 157. 1°. Let the magnitudes of the forces be $P_1$, $P_2$, &c., and the co-ordinates, with reference to the rectangular planes, of the points at which they act respectively, be $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, &c. Let also the direction cosines be $(l_1, m_1, n_1)$, $(l_2, m_2, n_2)$, &c. Resolve each force into three components, parallel to O X, O Y, O Z, respectively. Thus, if $(X_1, Y_1, Z_1)$, &c., be the components of $P_1$, &c., we shall have

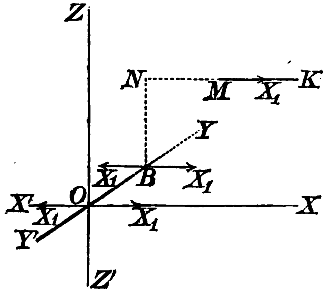
$$X_1 = P_1 l_1; \quad X_2 = P_2 l_2; \text{ \&c.} \quad . \quad . \quad . \quad . \quad . \quad (1),$$

$$Y_1 = P_1 m_1; \quad Y_2 = P_2 m_2; \text{ \&c.} \quad . \quad . \quad . \quad (2),$$

$$Z_1 = P_1 n_1; \quad Z_2 = P_2 n_2; \text{ \&c.} \quad . \quad . \quad . \quad . \quad . \quad (3).$$

To transfer these components to the point O. Let $X_1$, in M K (fig. 51) be the component, parallel to OX, of the force $P_1$ acting on the point M. From M transmit it along its line to a point N in the plane Z O Y: the co-ordinates of this point will be $y_1$, $z_1$. From N draw a perpendicular N B to O Y, and through B draw a line parallel to M K or O X. Introducing in this line a pair of balancing forces each equal to $X_1$, we

Fig. 51.



have a couple acting on an arm $z_1$ in a plane parallel to XOZ, and a single force $X_1$ parallel to OX in the plane XOY. The moment of this couple is $X_1 z_1$, and its axis is along OY. Next transfer the force $X_1$ from B to O, by introducing a pair of balancing forces in X′OX, one of which, with the force $X_1$ in the line through B parallel to X′X and the direction similar to OX, form a couple acting on an arm $y_1$. This couple, when Z and X are both positive, tends to turn in the plane XOY from OY towards OX. Therefore by the rule, § 40, its axis must be drawn from O in the direction OZ′. Hence its moment is to be reckoned as—$Z_1 x_1$. Besides this couple there remains a single force equal to $X_1$, in the direction OX, through the point O. Similarly by successive steps transfer the forces $Y_1$, $Z_1$, to the origin of co-ordinates. In this way six couples of transference are got, three tending to turn in one direction round the axes respectively, and three in the opposite direction; and three single forces at right angles to one another, acting at the point O. Thus for the force $P_1$, at the point $(x_1, y_1, z_1)$, we have as equivalent to it at the point O, three forces $X_1$, $Y_1$, $Z_1$, and three couples;

$Z_1 y_1 - Y_1 z_1$; moment of the couple round OX; . . . (4),

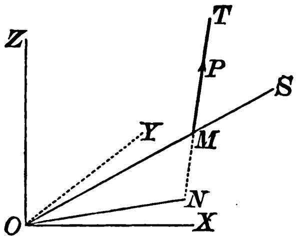
$X_1 z_1 - Z_1 x_1$; moment of the couple round OY; . . . (5),

$Y_1 x_1 - X_1 y_1$; moment of the couple round OZ. . . . (6).

All the forces may be brought in to the origin of co-ordinates in a similar way.

§ 158. 2°. Otherwise: Let P (fig. 52) be one of the forces acting in the line MT on a point M of a rigid body. Let O be the origin of co-ordinates; OX, OY, OZ, three rectangular lines of reference. Join OM and produce the line to S. From O draw ON, cutting at right angles in the point N, the line MT produced through M. Let ON be denoted by $p$, and the angle

Fig. 52.



T M S by $\kappa$. In a line through O parallel to M T (not shown in diagram) suppose introduced a pair of balancing forces each equal to P. We have thus a single force equal to P acting at O, and a couple, whose moment is P$p$, in the plane ONM. The direction cosines of this plane, or which is the same thing, the direction cosines of a perpendicular to it, that is, the axis of the couple are (§ 124), if we denote them by $\varphi$, $\chi$, $\psi$, respectively,

$$\varphi=\frac{\frac{y}{r}n-\frac{z}{r}m}{\sin\kappa},$$

$$\chi=\frac{\frac{z}{r}l-\frac{x}{r}n}{\sin\kappa},$$

$$\psi=\frac{\frac{x}{r}m-\frac{y}{r}l}{\sin\kappa}.$$

Now in the triangle O N M,

$$\mathrm{ON}=\mathrm{OM}\sin\mathrm{OMN},$$

that is

$$p=r\sin\kappa.$$

Hence, if we substitute $p$ for its value in the three preceding equations, the expression for the direction cosines are reduced to

$$\varphi=\frac{ny-mz}{p}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7),$$

$$\chi=\frac{lz-nx}{p}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8),$$

$$\psi=\frac{mx-ly}{p}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9).$$

To find the component couples round O X, O Y, O Z, multiply these direction cosines respectively by P$p$; whence we get

$\mathrm{P}\,p.\,\varphi=\mathrm{P}\,(ny-mz)$, moment of couple round O X, . . (10),
$\mathrm{P}\,p.\,\chi=\mathrm{P}\,(lz-nx)$, moment of couple round O Y, . . (11),
$\mathrm{P}\,p.\,\psi=\mathrm{P}\,(mx-ly)$, moment of couple round O Z. . . (12).

That this result is the same as that got by the other method will be evident, by considering that (Eq. 1, 2, 3),

$$\mathrm{P}\,l=\mathrm{X};\ \mathrm{P}\,m=\mathrm{Y};\ \mathrm{P}\,n=\mathrm{Z}.$$

§ 159. When by either of the methods all the forces have been referred to O, there is obtained a set of couples acting round O X, O Y, O Z; and a set of forces acting along O X, O Y, O Z. Find then the resultant moments of all the couples; and the sums of all the forces: if L, M, N, be the resultant moments round O X, O Y, O Z, respectively, we have

$$L = (Z_1 y_1 - Y_1 z_1) + (Z_2 y_2 - Y_2 z_2) + \&c. \quad . \quad . \quad (13),$$

$$M = (X_1 z_1 - Z_1 x_1) + (X_2 z_2 - Z_2 x_2) + \&c. \quad . \quad . \quad (14),$$

$$N = (Y_1 x_1 - X_1 y_1) + (Y_2 x_2 - X_2 y_2) + \&c. \quad . \quad . \quad (15);$$

and if X, Y, Z, be the resultant forces,

$$X = X_1 + X_2 + X_3 + \&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16),$$

$$Y = Y_1 + Y_2 + Y_3 + \&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17),$$

$$Z = Z_1 + Z_2 + Z_3 + \&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

§ 160. Finally, find the resultant of the three forces by the formulae of Chap. VI, and the resultant of the three couples by § 151, Prop. IV. Thus if $l, m, n$, be the direction cosines of the line of the resultant force R, we have (§ § 123, 130)

$$l = \frac{X}{R}; \ m = \frac{Y}{R}; \ n = \frac{Z}{R}, \quad . \quad . \quad . \quad (19),$$

and if $\lambda, \mu, \nu$, be the direction cosines of the axis of the resultant couple, we have (§ 151)

$$\lambda = \frac{L}{G}; \ \mu = \frac{M}{G}; \ \nu = \frac{N}{G}. \quad . \quad . \quad . \quad (20).$$

§ 161. Conditions of Equilibrium. The conditions of Equilibrium of three forces at right angles to one another have been already stated in § 133; and the conditions for three rectangular couples in § 152.

If a body be acted on by three forces and three couples simultaneously, all the conditions applicable when they act separately, must also be satisfied when they act conjointly, since a force cannot balance a couple. Six Equations of Equilibrium therefore are necessary and sufficient for a rigid body acted on by any number of forces. These are

$$P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \&c. = 0.$$
$$P_1 \cos \beta_1 + P_2 \cos \beta_2 + \&c. = 0.$$
$$P_1 \cos \gamma_1 + P_2 \cos \gamma_2 + \&c. = 0.$$
$$G_1 \cos \zeta_1 + G_2 \cos \zeta_2 + \&c. = 0.$$
$$G_1 \cos \eta_1 + G_2 \cos \eta_2 + \&c. = 0.$$
$$G_1 \cos \vartheta_1 + G_2 \cos \vartheta_2 + \&c. = 0.$$

§ 162. If the line of the resultant force found by § 160, is perpendicular to the plane of the couple, that is, if

$$\lambda = l, \ \mu = m, \ \nu \quad n;$$

or

$$\frac{L}{X} = \frac{M}{Y} = \frac{N}{Z}, \quad . \ . \ . \ . \ . \ . \quad (21),$$

the system cannot be reduced to another with a force and a smaller couple, and in this case the line found for the resultant force is the central axis of the system.

§ 163. If, on the other hand, the plane of the couple is parallel to the line of the force, or the axis of the couple perpendicular to the line of the force, that is, if

$$l \lambda + m \mu + n \nu = 0,$$

or

$$L X + M Y + N Z = 0, \quad . \ . \ . \ . \quad (22),$$

the force and couple may (§ 153) be reduced to one force: and this force is parallel to the former, at a distance from it equal to $\frac{G}{R}$, in the plane of it and the couple. Thus, (fig. 47) $O O'$ will be perpendicular to the line of the resultant force, and to the axis of the resultant couple, and therefore its direction cosines are (§ 124, *b*);

$$m \nu - n \mu, \ n \lambda - l \nu, \ l \mu - m \lambda, \quad . \ . \ . \ . \quad (23),$$

each of which will be positive when $O'$ lies within the solid angle edged by $O X$, $O Y$, $O Z$. Hence, remembering that $O O' = \frac{G}{R}$, and using the expressions (19) and (20), we find for the co-ordinates of $O'$

$$\frac{Y N - Z M}{R^2}, \ \frac{Z L - X N}{R^2}, \ \frac{X M - Y L}{R^2}, \quad . \ . \quad (24),$$

and we thus complete the specification of the single force to which the system is reduced when (22) holds.

§ 164. If the line of the force is inclined at any angle to the plane of the couple, the resultant system can be further reduced by § 154, to a smaller couple and a force in a determinate line, the "central axis." This couple is $G \cos \theta$, and according to the notation, may be thus expressed by § 124, (7), if we substitute the values given in (19) and (20),

$$G \cos \theta = \frac{XL + YM + ZN}{R}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (25).$$

The other component couple, $G \sin \theta$, lies in the same plane as R, and with it may be reduced by § 153 to one force, which will be parallel to R, that is, in the direction $(l, m, n)$, at a distance from it equal to $\frac{G \sin \theta}{R}$. Hence the direction cosines of $O O'$ will be

$$\frac{m\nu - n\mu}{\sin \theta}, \quad \frac{n\lambda - l\nu}{\sin \theta}, \quad \frac{l\mu - m\lambda}{\sin \theta}, \quad . \quad . \quad . \quad (26).$$

Substituting in each of these for $l$, $\lambda$, &c., their respective values, and multiplying each member by $\frac{G \sin \theta}{R}$, we have for the co-ordinates of the point $O'$, as in § 163,

$$\frac{YN - ZM}{R^2}, \quad \frac{ZL - XN}{R^2}, \quad \frac{XM - YL}{R^2}. \quad . \quad . \quad . \quad (27).$$

A single force, R, through the point thus specified in the direction $(l, m, n)$, with a couple in a plane perpendicular to it, and having

$$\frac{XL + YM + ZN}{R}$$

for its moment, is consequently the system of *force along central axis* and *minimum couple*, to which the given set of forces is determinately reducible by Poinsot's beautiful method.

§ 165. The position of the central axis may be determined otherwise; thus, instead of in the first place bringing the forces to O, bring them to any point T, of which let $(x, y, z)$ be the co-ordinates. Then instead of $Y_1 z_1 + Y_2 z_2 +$ &c., which we had before (§ 159), we have now

$$Y_1 (z_1 - z) + Y_2 (z_2 - z) + \&c.,$$

or

$$Y_1 z_1 + Y_2 z_2 + \&c. - (Y_1 + Y_2 + \&c.)\, z,$$

and so for the others. Then for the moments of the couples of transference we have

$$\mathfrak{L} = L - (Z y - Y z),$$
$$\mathfrak{M} = M - (X z - Z x),$$
$$\mathfrak{N} = N - (Y x - X y).$$

Now, let T be chosen, if possible, so as to make the resultant couple lie in a plane perpendicular to it. The condition to be fulfilled in this case is

$$\frac{\mathfrak{L}}{X} = \frac{\mathfrak{M}}{Y} = \frac{\mathfrak{N}}{Z},$$

which, when for $\mathfrak{L}$, &c., we substitute their values, becomes,

$$\frac{L - (Zy - Yz)}{X} = \frac{M - (Xz - Zx)}{Y} = \frac{N - (Yx - Xy)}{Z},$$

which is the equation of the central axis of the system.

To show that O$'$, the point determined in §§ 163, 164, is in the central axis thus found; we have, substituting for $x$, $y$, $z$, the values given in (24),

$$\frac{L - \dfrac{Z\,(Z L - X N) - Y\,(X M - Y L)}{R^2}}{X} = \&c.$$

Reducing, and remarking that

$$L R^2 - L Y^2 - L Z^2 = L X^2,$$

we find that the first member becomes

$$\frac{(L X + M Y + N Z)\, X}{X} = L X + M Y + N Z,$$

and is therefore equal to each of the two others. Thus is verified the comparison of the two methods.

§ 166. In one respect, this reduction of a system of forces to a couple, and a force perpendicular to its plane, is the best and simplest, especially in having the advantage of being determinate, and it gives very clear and useful conceptions regarding the effect of force on a rigid body. The system may, however, be farther reduced to two equal forces acting symmetrically on the rigid body, but whose position is indeterminate. Thus, supposing the central axis of the system has been found, draw a line $A A'$, at right angles through any point $C$ in it, so that $C A$ may be equal to $C A'$. For $R$, acting along the central axis, substitute $\frac{1}{2} R$ at each end of $A A'$. Thus, choosing this line $A A'$ as the arm of the couple, and calling it $a$, we have at each extremity of it, two forces, $\frac{G}{a}$ perpendicular to the central axis, and $\frac{1}{2} R$ parallel to the central axis. Compounding these we get two forces, each equal to $(\frac{1}{4} R^2 + \frac{G^2}{a^2})^{\frac{1}{2}}$, through $A$ and $A'$ respectively, perpendicular to $A A'$, and equally inclined at the angle $\tan^{-1} \frac{2 G}{a R}$ on the two sides of the plane through $A A'$ and the central axis.

# APPENDIX.

# APPENDIX.[*]

## ON THE POTENTIAL OF A CLOSED GALVANIC CIRCUIT OF ANY FORM.[†]

THE object of the following note is to point out an extremely interesting application of the principles explained by Professor De Morgan in the preceding paper,[‡] which occurred to me in connection with the determination of the potential of an electro-magnet in terms of the solid angle of a cone.

It has been shewn by Ampère that a *closed galvanic circuit in a re-entering curve of any form* produces the same magnetic action as any infinitely thin sheet of steel having this curve for its edge, would produce if uniformly and normally magnetized. Now the resultant force of a magnet at any point may be expressed, after the manner of Laplace, in terms of the differential coefficients of a "potential function," and therefore the same proposition is true for a closed galvanic circuit.[||] When this is known to be true, for either a common or an electro-magnet, the following definition may be laid down.

The potential at any point in the neighbourhood of a magnet is the quantity of work necessary to bring a unit north-pole (or the north-pole of an infinitely thin uniformly and longitudinally magnetized unit-bar) from an infinite distance to that point. To determine the potential at any point due to a given closed galvanic circuit, let us imagine a magnetized sheet of steel (the form of the sheet is arbitrary, provided only that its edge coincide with the curve of the galvanic circuit), which according to Ampère produces the same magnetic action, and consequently the same potential as a given closed circuit, to be divided into infinitely small areas. Then it is easily demonstrated, on the most elementary principles of the theory of magnetism, that the potentials at any point, P, produced by these areas are proportional to the solid angles which they subtend at P ; the true *sign* of the potential of any small area being obtained by considering the solid angle as positive, if the side of the area containing north poles, or negative, if the other side be towards P. Hence the potential of the whole sheet of steel, at any point P, is proportional to the entire solid angle which it subtends at P ; and consequently the potential of a closed galvanic circuit, at any point P, is equal to a constant (which may be taken as a measure of the strength of the galvanism, or as it is often termed, the "quantity" of the current) multiplied into the solid angle of the cone described by a straight line always passing through P, and carried round the circuit. In all cases except those in which the galvanic circuit is contained in one plane, there will be positions of P for which this cone will be "autotomic ;" and in many cases, especially the most common practical case of an electro-magnet, in which the circuit consists of double or multiple concentric helices with their ends connected, or of a single wire wrapped in a complex manner round a body of some irregular shape, so as to constitute most complicated curves of double curvature, there will be no position of the point P for which the cone is not excessively autotomic. The solid angle of such a cone or the area enclosed by its intersection with a spherical surface of unit radius, having for centre its vertex, may be determined in a manner precisely similar to that which has been explained by Professor de Morgan for plane self-cutting curves, without any ambiguity as to the *circuit* by which the curve, when self-cutting,* is to be described, since the actual galvanic current is in a determinate circuit, and its projection, by the conical surface, on the surface of the sphere is to be described by the projection of a point moving along the electric conductor, either in the same directlon as the current, or in the opposite, according to the convention we please to make. There is, however, a source of ambiguity which really affects the evaluation of the solid angle of a cone, or of the area of any given circuit described in a determinate manner on a spherical surface and gives rise to a multiplicity of solutions of the problem arising from the circumstance that of all the "primary parts"† (only two in number of the circuit be not self-cutting) into which the spherical surface is divided by the curve, there is no reason for choosing one more than another as a zero space (or a space corresponding to a space exterior to a closed circuit in a plane).*

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[*] See note †, page 45.

[†] [Reprinted from the Cambridge and Dublin Mathematical Journal, 1850.]

[‡] [Namely, *On the Extension of the word Area*, referred to in § 104 and the note †.]

[||] In other words, the quantity of *work* necessary to bring a magnetic pole from any position in the neighbourhood of a closed galvanic circuit to any other position does not *vary* with the form of the curve along which it is drawn from one point to the other. There is, however, one remarkable difference between the case of an electro-magnet and that of any given steel magnet. In the case of an electro-magnet, although the quantity of work does not *vary* with the path, yet it has determinately different values according as the path lies on one side, or on another of any part of the galvanic wire circuit, or according to the convolutions round any part of the wire which it may be arbitrarily chosen to make. Hence arises the multiplicity of values of the potential at any point in the neighbourhood of an electro-magnet noticed below. Yet for any one form of a magnetized sheet of steel of the kind described in the text, agreeing, in the action which it produces on all points not in its own substance, with the electro-magnet, the potential is perfectly determinate without a multiplicity of values; and the difference in the two cases is accounted for when we consider that the magnetic potentials at any two points infinitely near one another, on the two sides of the sheet of steel, differ by $4\pi\gamma$ where $\gamma$ is a constant such that $\gamma\omega$ is the magnetic moment of any infinitely small area $\omega$ of the sheet. The agreement in the magnetic circumstances in the two cases fails for all points in the substance of the magnetized steel.

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* [See note on the word "circuit" to § 103.]

† [Any one portion in which we can get from one point to another without crossing the boundary is defined as a primary part, by De Morgan.]

When the value of the area, according to any one of these solutions, has been obtained, all the others may be deduced, by adding to it or subtracting from it any number of times the area of the whole spherical surface. Hence the most general expression for the solid angle of a cone described in a determinate manner, is

$$\sigma = \sigma_1 + 4 i \pi,$$

where $\sigma_1$ denotes any one value and $i$ any positive or negative integer. If too great a positive or too small a negative value be given to $i$ all the "primary spaces" of the spherical surface will be positive or all will be negative; and therefore if we wish to obtain only those solutions according to which some portion of the spherical surface is considered as *zero* or *external to the circuit*, a limited number only (not exceeding the number of primary parts into which the spherical surface is divided by the circuit) of values for $i$ are to be admitted. The physical problem, however, requires no limitation to the range of values that may be given to $i$: for if we take any two paths to the point P from an infinite distance, such that the space between them is *once* crossed by a galvanic circuit, the potential at P will differ by $4 \pi \gamma$ according as it is estimated by one path or by the other; and therefore, by taking (for the sake of simplicity in the conception) different paths to the point P which go round a certain portion of the galvanic circuit once, twice, three times, four times, &c., in one direction, and again different paths which go round the same portion of the wire once, twice, three times, four times, &c. in the contrary direction, we obtain, according to the definition, an infinite number of values of the potential at the point P, which are successively expressed by the formula

$$v = v_1 + 4 i \pi \gamma,$$

when we give $i$ the values, 1, 2, 3, 4, &c., and again the values —1, —2, —3, —4, &c.; $v_1$ being the potential estimated by a path, which makes none of the convolutions of the kind described with reference to the others.

Hence we see that, to find the general expression for the potential at a point in the neighbourhood of an electro-magnet, we may first choose some determinate path from an infinite distance to the point P, and investigate the value of the potential for it, which may be used as the value of

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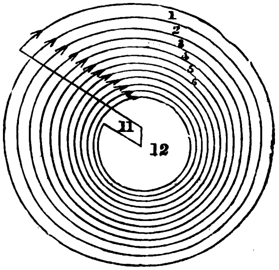
* Thus, if the given curve be a circle of the sphere, described in a given direction, and if $\theta$ denote the angular radius measured from that pole O, which would be *north* if the direction of describing the circle were from west to east; the area of the circuit is $2 \pi (1 - \cos \theta)$ if the space on the other side of the circle from O be considered as the zero space, but it would be $-2 \pi (1 + \cos \theta)$ if the space in which O is situated were taken as zero, or external to the circuit. In general, the area of a circuit not self-cutting, on a spherical surface, will be either, one of the two parts into which the spherical surface is divided, with the sign +, or the other part, with the sign —.

$v_1$ in the preceding expression. If an infinite straight line in any direction, terminated at the point P, be the path chosen, the determinate potential will be found by considering the primary portion of the spherical surface described from P as centre, which is cut by this line, as the portion external to the circuit. Hence, if we mark this primary portion with a zero, the number with which any other primary part is to be marked, according to Professor De Morgan's rule, will be got by drawing a line to any point within it, from any point O, in the external primary part, and counting the number of times it is cut by the curve; every time it is cut from right to left (with reference to a person walking from O, along it, on the convex surface of the sphere) being counted as + 1, and every time it is cut in the other direction, as — 1, and the algebraical sum taken. When the number for each primary part has been thus determined, the sum of the areas, each multiplied by its number, (positive or negative, as the case may be) of the different primary parts, will be the required area of the circuit; and the potential at the centre of the sphere will be obtained by multiplying this by $\gamma$, the strength of the galvanic current. The absolute sign of the potential thus determined may be readily shown to be correct, if we agree to consider the potential due to terrestrial magnetism as on the whole positive for positions north, and negative for positions south of the magnetic equator, since, as is well known, currents round the earth, proceeding on the whole from east to west, would produce phenomena similar to the actual phenomena of terrestrial magnetism.

As an example, let us consider a conducting circuit which consists of twelve complete spires of a helix, and a line along the axis with two perpendicular portions connecting its extremities with those of the helix. The accompanying diagrams represent the projections, by radii, of the circuit, on a spherical surface in two different positions, viewed in each case from the interior of the sphere.

In the case illustrated by fig. (1), the centre of the sphere is nearly in a line with the axis of the helix on the side towards the north pole*

Fig. 1.



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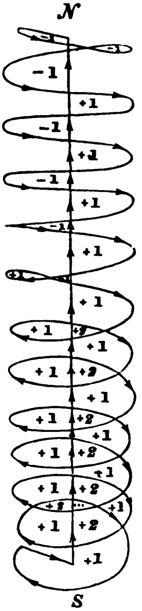
[*] The ends of the helix which would be repelled from the north and from the south respectively by the earth's magnetic action are, in the ordinary vague use of the term "pole," called the north and south poles of the electro-magnet.

of the helix, and distant from it by about half the length of the axis. In the case illustrated by fig. (2), the centre of the sphere is in a perpendicular through a point of the axis, distant by about one-fourth of its length from the north pole of the helix, and is at about the same distance from the nearest part of the helix, as in the case of fig. (1); and the curve on the spherical surface is shown in the diagram, according to Mercator's projection with the great circle containing the axis of the helix as equator.* In each diagram the inner side of the spherical surface is shown.

Fig. 2.



The radii of the spheres being supposed to be equal in the two cases, if we denote their common value by $r$, and if $A_1$ and $A_2$ be the areas of the spherical curves, represented in the diagrams, the zero or external portions on the spherical surfaces being taken as those which become infinite in the plane diagrams, the values of the potential at the centre of the sphere will be—

$$\gamma \frac{A_1}{r^2}, \text{ and } \gamma \frac{A_2}{r^2},$$

respectively, for any paths from an infinite distance which do not lie round any portion of the galvanic wire, nor between any of the spires.

The area $A_1$ will be determined (in accordance with Professor De Morgan's rule†) by finding the areas of the "primary parts," marked successively with the numbers 1, 2, ... up to 12, multiplying each area by the corresponding number, and taking the sum of the products. The area $A_2$ will be similarly determined by finding the areas of the primary parts in fig. (2), multiplying each by the positive or negative number with which it is marked, and taking the algebraic sum of the products.

GLASGOW COLLEGE, *March* 25, 1850.

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* The diagram was actually drawn by tracing the shadow of a helix of twelve spires, ¾ in. in diameter and 4 in. in length, upon a cylindrical surface, produced by a luminous point at its centre; the axis of the helix being held in a plane through the luminous point at right angles to the axis of the cylindrical surface. On account of the narrowness of the band occupied by the diagram, the cylindrical surface very nearly coincided with the spherical surface, which in strictness ought to have received the shadow. After the shadow was thus traced, the cylindrical surface was unbent into a plane.

† In fig. (1), all the arrow-heads which are necessary for rendering determinate the "balances" for the primary parts are given; and the numbers expressing the balances are marked for the first six primary parts, commencing with the outermost. In fig. (2), all the arrow-heads which are necessary to make the diagram represent determinately a closed circuit are indicated, except in a few places where the spaces are too confined for admitting of this being done in a clear manner; and the "balances" of all the primary parts are marked with numbers, excepting in the instance of a very small *triple* primary part, which is marked with three dots (...) instead of + 3.

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